CSE 135: Introduction to Theory of Computation
Equivalence of DFA and NFA

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Expressive Power of NFAs and DFAs

- Is there a language that is recognized by a DFA but not by any NFAs?
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- Is there a language that is recognized by an NFA but not by any DFAs? No!
Main Theorem

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A language $L$ is regular if and only if there is an NFA $N$ such that $L(N) = L$. 
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A language $L$ is regular if and only if there is an NFA $N$ such that $L(N) = L$.
In other words:
- For any DFA $D$, there is an NFA $N$ such that $L(N) = L(D)$, and
- for any NFA $N$, there is a DFA $D$ such that $L(D) = L(N)$.
Converting DFAs to NFAs

Proposition

For any DFA $D$, there is an NFA $N$ such that $L(N) = L(D)$.
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Proof.

Is a DFA an NFA?
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Is a DFA an NFA? Essentially yes! Syntactically, not quite. The formal definition of DFA has $\delta_{\text{DFA}} : Q \times \Sigma \rightarrow Q$ whereas $\delta_{\text{NFA}} : Q \times (\Sigma \cup \{\epsilon\}) \rightarrow \mathcal{P}(Q)$.  

□
Converting DFAs to NFAs

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For DFA $D = (Q, \Sigma, \delta_D, q_0, F)$, define an “equivalent” NFA $N = (Q, \Sigma, \delta_N, q_0, F)$ that has the exact same set of states, initial state and final states. Only difference is in the transition function.

$$\delta_N(q, a) = \{\delta_D(q, a)\}$$

for $a \in \Sigma$ and $\delta_N(q, \epsilon) = \emptyset$ for all $q \in Q$.  \(\square\)
Simulating an NFA on Your Computer

NFA Acceptance Problem
Given an NFA $N$ and an input string $w$, does $N$ accept $w$?
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Given an NFA $N$ and an input string $w$, does $N$ accept $w$?

How do we write a computer program to solve the NFA Acceptance problem?
Two Views of Nondeterminism

**Guessing View**
At each step, the NFA “guesses” one of the choices available; the NFA will guess an “accepting sequence of choices” if such a one exists.

**Parallel View**
At each step the machine “forks” threads corresponding to each of the possible next states.
Two Views of Nondeterminism

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At each step, the NFA “guesses” one of the choices available; the NFA will guess an “accepting sequence of choices” if such a one exists.
Very useful in reasoning about NFAs and in designing NFAs.

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At each step the machine “forks” threads corresponding to each of the possible next states.
Two Views of Nondeterminism

Guessing View
At each step, the NFA “guesses” one of the choices available; the NFA will guess an “accepting sequence of choices” if such a one exists.
Very useful in reasoning about NFAs and in designing NFAs.

Parallel View
At each step the machine “forks” threads corresponding to each of the possible next states.
Very useful in simulating/running NFA on inputs.
Algorithm for Simulating an NFA

Algorithm
Keep track of the current state of each of the active threads.
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Keep track of the current state of each of the active threads.

Example

Consider the input $w = 111$. The execution (listing only the states of currently active threads) is

Example NFA $N$
**Algorithm for Simulating an NFA**

**Algorithm**

Keep track of the current state of each of the active threads.

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Consider the input $w = 111$. The execution (listing only the states of currently active threads) is

\[ \langle q_0 \rangle \]
Algorithm for Simulating an NFA

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Keep track of the current state of each of the active threads.

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Consider the input $w = 111$. The execution (listing only the states of currently active threads) is

$$\langle q_0 \rangle \xrightarrow{1} \langle q_0, q_1 \rangle$$
Algorithm for Simulating an NFA

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Keep track of the current state of each of the active threads.

Example
Consider the input $w = 111$. The execution (listing only the states of currently active threads) is

$$
\langle q_0 \rangle \xrightarrow{1} \langle q_0, q_1 \rangle \xrightarrow{1} \langle q_0, q_1, q_1 \rangle \xrightarrow{1} \langle q_0, q_1, q_1, q_1 \rangle
$$
Observations

- Exponentially growing memory: more threads for longer inputs. Can we do better?
Algorithm
With optimizations

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▶ Exact order of threads is not important
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  - It is unimportant whether the 5th thread or the 1st thread is in state q.
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- If two threads are in the same state, then we can ignore one of the threads
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Observations

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▶ Exact order of threads is not important
  ▶ It is unimportant whether the 5th thread or the 1st thread is in state q.
▶ If two threads are in the same state, then we can ignore one of the threads
  ▶ Threads in the same state will “behave” identically; either one of the descendent threads of both will reach a final state, or none of the descendent threads of both will reach a final state
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$$\{q_0\}$$
Parsimonious Algorithm in Action

Example

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$$\{q_0\} \xrightarrow{1} \{q_0, q_1\}$$
Example

Example NFA $N$

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Revisiting NFA Simulation Algorithm

- Need to keep track of the states of the active threads
  - Unordered: Without worrying about exactly which thread is in what state
  - No Duplicates: Keeping only one copy if there are multiple threads in same state
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- How much memory is needed?

If $Q$ is the set of states of the NFA $N$, then we need to keep a subset of $Q!$ states, which is finite!!
Revisiting NFA Simulation Algorithm

- Need to keep track of the states of the active threads
  - **Unordered**: Without worrying about exactly which thread is in what state
  - **No Duplicates**: Keeping only one copy if there are multiple threads in same state

- How much memory is needed?
  - If $Q$ is the set of states of the NFA $N$, then we need to keep a subset of $Q$!
  - Can be done in $|Q|$ bits of memory (i.e., $2^{|Q|}$ states), which is finite!!
Constructing an Equivalent DFA

- The DFA runs the simulation algorithm

- DFA remembers the current states of active threads without duplicates, i.e., maintains a subset of states of the NFA

- When a new symbol is read, it updates the states of the active threads

- Accepts whenever one of the threads is in a final state
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Example of Equivalent DFA (1)

Example NFA $N$

DFA equivalent to $N$
Example of Equivalent DFA (1)

Example NFA $N$

Example DFA $D$ equivalent to $N$
Example of Equivalent DFA (1)

Example NFA $N$

DFA $D$ equivalent to $N$

We can remove states unreachable from the initial state. How to find such states?

Use DFS or BFS!
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We can remove states unreachable from the initial state. How to find such states? Use DFS or BFS!
Example of Equivalent DFA (2)

Example NFA $N_\varepsilon$
Example of Equivalent DFA (2)

Example NFA $N_\epsilon$

DFA $D'_\epsilon$ for $N_\epsilon$ (only relevant states)
Example of Equivalent DFA (3)

Example NFA $N$

![Diagram of NFA and DFA](image)
Example of Equivalent DFA (3)

Example NFA $N$

DFA $D$ equivalent to $N$
Given NFA $N = (Q, \Sigma, \delta, q_0, F)$, construct DFA $\text{det}(N) = (Q', \Sigma, \delta', q_0', F')$ as follows.

- $Q' =$
- $q_0' =$
- $F' =$
Given NFA $N = (Q, \Sigma, \delta, q_0, F)$, construct DFA $\text{det}(N) = (Q', \Sigma, \delta', q'_0, F')$ as follows.

1. $Q' = \mathcal{P}(Q)$
2. $q'_0 = \hat{\Delta}(q_0, \varepsilon)$
3. $F' = \{ A \subseteq Q | A \cap F \neq \emptyset \}$
4. $\delta'(A, a) = \hat{\Delta}(q_1, a) \cup \hat{\Delta}(q_2, a) \cup \cdots \cup \hat{\Delta}(q_k, a)$ or more concisely, $\delta'(A, a) = \bigcup_{q \in A} \hat{\Delta}(q, a)$.
Formal Construction

Given NFA \( N = (Q, \Sigma, \delta, q_0, F) \), construct DFA \( \text{det}(N) = (Q', \Sigma, \delta', q'_0, F') \) as follows.

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  \[
  \delta'(A, a) = \bigcup_{q \in A} \hat{\Delta}(q, a)
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Correctness

Lemma

For any NFA $N$, the DFA $\text{det}(N)$ is equivalent to it, i.e., $L(N) = L(\text{det}(N))$. 
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$$\forall w \in \Sigma^*. \text{det}(N) \text{ accepts } w \text{ iff } N \text{ accepts } w$$
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Need to show

\[
\forall w \in \Sigma^*. \quad \text{det}(N) \text{ accepts } w \iff N \text{ accepts } w
\]

\[
\forall w \in \Sigma^*. \quad \hat{\delta}(q'_0, w) \in F' \iff \hat{\Delta}(q_0, w) \cap F \neq \emptyset
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For any NFA $N$, the DFA $\text{det}(N)$ is equivalent to it, i.e., $L(N) = L(\text{det}(N))$.

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$$\forall w \in \Sigma^*. \hat{\delta}(q'_0, w) \in F' \text{ iff } \hat{\Delta}(q_0, w) \cap F \neq \emptyset$$
$$\forall w \in \Sigma^*. \text{ for } A = \hat{\delta}(q'_0, w), A \cap F \neq \emptyset \text{ iff } \hat{\Delta}(q_0, w) \cap F \neq \emptyset$$
Correctness

Lemma

For any NFA $N$, the DFA $\text{det}(N)$ is equivalent to it, i.e., $L(N) = L(\text{det}(N))$.

Proof Idea

Need to show

$\forall w \in \Sigma^*. \text{det}(N)$ accepts $w$ iff $N$ accepts $w$

$\forall w \in \Sigma^*. \hat{\delta}(q_0', w) \in F' \text{ iff } \hat{\Delta}(q_0, w) \cap F \neq \emptyset$

$\forall w \in \Sigma^*. \text{ for } A = \hat{\delta}(q_0', w), \ A \cap F \neq \emptyset \text{ iff } \hat{\Delta}(q_0, w) \cap F \neq \emptyset$

We will instead prove the stronger claim $\forall w \in \Sigma^*. \hat{\delta}(q_0', w) = A \text{ iff } \hat{\Delta}(q_0, w) = A$. 
Lemma
\[ \forall w \in \Sigma^*. \delta(q_0', w) = A \iff \Delta(q_0, w) = A. \]
**Correctness Proof**

**Lemma**
\[ \forall w \in \Sigma^*. \hat{\delta}(q'_0, w) = A \iff \hat{\Delta}(q_0, w) = A. \]

**Proof.**

By induction on \(|w|\)
Correctness Proof

Lemma
\[ \forall w \in \Sigma^*. \hat{\delta}(q'_0, w) = A \iff \hat{\Delta}(q_0, w) = A. \]

Proof.
By induction on \(|w|\)
   - Base Case \(|w| = 0\): Then \(w = \epsilon\). Now
Correctness Proof

Lemma
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Proof.
By induction on \(|w|\)
   
   - Base Case \(|w| = 0\): Then \(w = \epsilon\). Now
     \[
     \hat{\delta}(q_0', \epsilon) = q_0' \quad \text{defn. of } \hat{\delta}
     \]
Correctness Proof

Lemma
\[ \forall w \in \Sigma^*. \hat{\delta}(q'_0, w) = A \iff \hat{\Delta}(q_0, w) = A. \]

Proof.
By induction on \(|w|\)
- **Base Case** \(|w| = 0\): Then \(w = \epsilon\). Now

\[
\hat{\delta}(q'_0, \epsilon) = q'_0 \\
= \hat{\Delta}(q_0, \epsilon) \\
\text{defn. of } \hat{\delta} \\
\text{defn. of } q'_0
\]
Correctness Proof

Lemma
\( \forall w \in \Sigma^*. \ \hat{\delta}(q'_0, w) = A \iff \hat{\Delta}(q_0, w) = A. \)

Proof.
By induction on \(|w|\)

- **Base Case** \(|w| = 0\): Then \(w = \epsilon\). Now

\[
\hat{\delta}(q'_0, \epsilon) = q'_0 \quad \text{defn. of } \hat{\delta}
\]
\[
= \hat{\Delta}(q_0, \epsilon) \quad \text{defn. of } q'_0
\]

- **Induction Hypothesis**: Assume inductively that the statement holds \(\forall w. \ |w| = n\)

\[
\Rightarrow \quad \hat{\delta}(q'_0, w) = A \iff \hat{\Delta}(q_0, w) = A
\]
Correctness Proof

Induction Step

Proof (contd).

- **Induction Step:** If $|w| = n + 1$ then $w = ua$ with $|u| = n$ and $a \in \Sigma$.

$$\hat{\delta}(q_0', ua) = \delta(\hat{\delta}(q_0', u), a)$$

defn. of $\hat{\delta}$
Proof (contd).

- **Induction Step:** If $|w| = n + 1$ then $w = ua$ with $|u| = n$ and $a \in \Sigma$.

\[
\hat{\delta}(q'_0, ua) = \delta(\hat{\delta}(q'_0, u), a) \quad \text{defn. of}\ \hat{\delta}
\]
\[
= \delta(\hat{\Delta}(q_0, u), a) \quad \text{ind. hyp.}
\]
**Correctness Proof**

*Induction Step*

Proof (contd).

- **Induction Step**: If $|w| = n + 1$ then $w = ua$ with $|u| = n$ and $a \in \Sigma$.

\[
\hat{\delta}(q'_0, ua) = \delta(\hat{\delta}(q'_0, u), a) = \delta(\hat{\Delta}(q_0, u), a) = \bigcup_{q \in \hat{\Delta}(q_0, u)} \hat{\Delta}(q, a)
\]

\[
\text{defn. of } \hat{\delta} \quad \text{ind. hyp.} \quad \text{defn. of } \delta
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Correctness Proof
Induction Step

Proof (contd).

- **Induction Step**: If $|w| = n + 1$ then $w = ua$ with $|u| = n$ and $a \in \Sigma$.

\[
\hat{\delta}(q'_0, ua) = \delta(\hat{\delta}(q'_0, u), a) \\
= \delta(\hat{\Delta}(q_0, u), a) \quad \text{ind. hyp.} \\
= \bigcup_{q \in \hat{\Delta}(q_0, u)} \hat{\Delta}(q, a) \\
= \hat{\Delta}(q_0, ua) \quad \text{prop. about } \hat{\Delta}
\]