CSE 135: Introduction to Theory of Computation
Variants of Turing Machines and Church-Turing Thesis

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Multi-Tape Turing Machine

finite-state control

0 0 1 □ □

1 0 □ 0 0 □

0 1 1 0 □ □

Input on Tape 1

Initially all heads scanning cell 1, and tapes 2 to k blank

In one step: Read symbols under each of the k heads, and depending on the current control state, write new symbols on the tapes, move the each tape head (possibly in different directions), and change state.
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A $k$-tape Turing Machine is $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$ where

- $Q$ is a finite set of control states
- $\Sigma$ is a finite set of input symbols
- $\Gamma \supseteq \Sigma$ is a finite set of tape symbols. Also, a blank symbol $\mathbb{0} \in \Gamma \setminus \Sigma$
- $q_0 \in Q$ is the initial state
- $q_{\text{acc}} \in Q$ is the accept state
- $q_{\text{rej}} \in Q$ is the reject state, where $q_{\text{rej}} \neq q_{\text{acc}}$
- $\delta : (Q \setminus \{q_{\text{acc}}, q_{\text{rej}}\}) \times \Gamma_k \rightarrow Q \times (\Gamma \times \{L, R\})_k$ is the transition function.
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Computation, Acceptance and Language

- A configuration of a multi-tape TM must describe the state, contents of all \( k \)-tapes, and positions of all \( k \)-heads. Thus, \( c \in Q \times (\Gamma^*\{\ast}\Gamma\Gamma^*)^k \), where \( \ast \) denotes the head position.
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\( L(M) = \{w | w \text{ accepted by } M\} \)
Expressive Power of multi-tape TM

Theorem
For any k-tape Turing Machine M, there is a single tape TM \( \text{single}(M) \) such that \( L(\text{single}(M)) = L(M) \).

Challenges

▶ How do we store \( k \)-tapes in one?
▶ How do we simulate the movement of \( k \) independent heads?
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Store in cell $i$ contents of cell $i$ of all tapes.
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1-tape equivalent single($M$)
Simulating One Step

**Challenge 1:** Head of 1-Tape TM is pointing to one cell. How do we find out all the $k$ symbols that are being read by the $k$ heads, which maybe in different cells?

▶ Read the tape from left to right, storing the contents of the cells being scanned in the state, as we encounter them.

**Challenge 2:** After this scan, 1-tape TM knows the next step of $k$-tape TM. How do we change the contents and move the heads?

▶ Once again, scan the tape, change all relevant contents, “move” heads (i.e., move $\ast$), and change state.
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1. First the machine will rewrite input $w$ to be in “new” format.
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   - Read from left-to-right, changing symbols, and moving those heads that need to be moved right.
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   - Read from left-to-right, changing symbols, and moving those heads that need to be moved right.
   - Scan back from right-to-left moving the heads that need to be moved left.
Nondeterministic Turing Machine

**Deterministic TM:** At each step, there is one possible next state, symbols to be written and direction to move the head, or the TM may halt.

**Nondeterministic TM:** At each step, there are finitely many possibilities. So formally, $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$, where $Q$, $\Sigma$, $\Gamma$, $q_0$, $q_{\text{acc}}$, $q_{\text{rej}}$ are as before for 1-tape machine $\delta : (Q \setminus \{q_{\text{acc}}, q_{\text{rej}}\}) \times \Gamma \to P(Q \times \Gamma \times \{\text{L}, \text{R}\})$.
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A single step $\vdash$ is defined similarly.

$X_1 X_2 \cdots X_{i-1} q X_i \cdots X_n \vdash X_1 X_2 \cdots p X_{i-1} Y \cdots X_n$, if $(p, Y, L) \in \delta(q, X_i)$
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Theorem

For any nondeterministic Turing Machine $M$, there is a (deterministic) TM $\text{det}(M)$ such that $L(\text{det}(M)) = L(M)$.

Proof Idea

$\text{det}(M)$ will simulate $M$ on the input.

- Idea 1: $\text{det}(M)$ tries to keep track of all possible "configurations" that $M$ could possibly be after each step. Works for DFA simulation of NFA but not convenient here.

- Idea 2: $\text{det}(M)$ will simulate $M$ on each possible sequence of computation steps that $M$ may try in each step.
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Nondeterministic Computation

- If $r = \max_{q,X} |\delta(q, X)|$ then the runs of $M$ can be organized as an $r$-branching tree.
Nondeterministic Computation

\[ C_\epsilon = q_0 w \]

\[ C_1 \quad \ldots \quad C_i \quad \ldots \quad \ldots \quad \ldots \quad C_r \]

\[ \ldots \quad \ldots \quad C_{ij} \quad \ldots \quad C_{r1} \quad \ldots \quad C_{rr} \]

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Input \( w \) is accepted iff \( \exists \) accepting configuration in tree.
Proof Idea

The machine $\text{det}(M)$ will search for an accepting configuration in computation tree.
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The machine det($M$) will search for an accepting configuration in computation tree

- The configuration at any vertex can be obtained by simulating $M$ on the appropriate sequence of nondeterministic choices

Why not a DFS? Observe that det($M$) may not terminate if $w$ is not accepted.
The machine $\text{det}(M)$ will search for an accepting configuration in computation tree

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- $\text{det}(M)$ will perform a BFS on the tree.
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- Tape 1, called input tape, will always hold input $w$
- Tape 2, called simulation tape, will be used as $M$’s tape when simulating $M$ on a sequence of nondeterministic choices
Proof Details

det(M) will use 3 tapes to simulate M (note, multitape TMs are equivalent to 1-tape TMs)

- Tape 1, called input tape, will always hold input w
- Tape 2, called simulation tape, will be used as M’s tape when simulating M on a sequence of nondeterministic choices
- Tape 3, called choice tape, will store the current sequence of nondeterministic choices
Execution of $\text{det}(M)$

1. Initially: Input tape contains $w$, simulation tape and choice tape are blank
2. Copy contents of input tape onto simulation tape
3. Simulate $M$ using simulation tape as its (only) tape
   3.1 At the next step of $M$, if state is $q$, simulation tape head reads $X$, and choice tape head reads $i$, then simulate the $i$th possibility in $\delta(q, X)$; if $i$ is not a valid choice, then goto step 4
   3.2 After changing state, simulation tape contents, and head position on simulation tape, move choice tape’s head to the right. If Tape 3 is now scanning $\Box$, then goto step 4
   3.3 If $M$ accepts then accept and halt, else goto step 3(1) to simulate the next step of $M$.
4. Write the lexicographically next choice sequence on choice tape, erase everything on simulation tape and goto step 2.
Deterministic Simulation
In a nutshell

▶ det($M$) simulates $M$ over and over again, for different sequences, and for different number of steps.
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- $\text{det}(M)$ simulates $M$ over and over again, for different sequences, and for different number of steps.
- If $M$ accepts $w$ then there is a sequence of choices that will lead to acceptance. $\text{det}(M)$ will eventually have this sequence on choice tape, and then its simulation $M$ will accept.
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- det($M$) simulates $M$ over and over again, for different sequences, and for different number of steps.
- If $M$ accepts $w$ then there is a sequence of choices that will lead to acceptance. det($M$) will eventually have this sequence on choice tape, and then its simulation $M$ will accept.
- If $M$ does not accept $w$ then no sequence of choices leads to acceptance. det($M$) will therefore never halt!
Random Access Machines

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- Initially, the program instructions are stored in a contiguous block of memory locations starting at location 1. All registers and memory locations, other than those storing the program, are set to 0.
Instruction Set

- **add X, Y**: Add the contents of registers X and Y and store the result in X.
- **loadc X, I**: Place the constant I in register X.
- **load X, M**: Load the contents of memory location M into register X.
- **loadI X, M**: Load the contents of the location “pointed to” by the contents of M into register X.
- **store X, M**: Store the contents of register X in memory location M.
- **jmp M**: The next instruction to be executed is in location M.
- **jmz X, M**: If register X is 0, then jump to instruction M.
- **halt**: Halt execution.
Expressive Power of RAMs

Theorem

Anything computed on a RAM can be computed on a Turing machine.
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Robustness of the Class of TM Languages

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- Non-Turing Machine models: random access machines, \( \lambda \)-calculus, type 0 grammars, first-order reasoning, \( \pi \)-calculus, ...
Various efforts to capture mechanical computation have the same expressive power.

- Non-Turing Machine models: random access machines, λ-calculus, type 0 grammars, first-order reasoning, π-calculus, ...

- Enhanced Turing Machine models: TM with 2-way infinite tape, multi-tape TM, nondeterministic TM, probabilistic Turing Machines, quantum Turing Machines ...
Various efforts to capture mechanical computation have the same expressive power.

- Non-Turing Machine models: random access machines, $\lambda$-calculus, type 0 grammars, first-order reasoning, $\pi$-calculus, ...
- Enhanced Turing Machine models: TM with 2-way infinite tape, multi-tape TM, nondeterministic TM, probabilistic Turing Machines, quantum Turing Machines ...
- Restricted Turing Machine models: queue machines, 2-stack machines, 2-counter machines, ...
Church-Turing Thesis

“Anything solvable via a mechanical procedure can be solved on a Turing Machine.”
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- Not a mathematical statement that can be proved or disproved!
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“Anything solvable via a mechanical procedure can be solved on a Turing Machine.”

- Not a mathematical statement that can be proved or disproved!
- Strong evidence based on the fact that many attempts to define computation yield the same expressive power
Consequences

- In the course, we will use an informal pseudo-code to argue that a problem/language can be solved on Turing machines.
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- To show that something can be solved on Turing machines, you can use any programming language of choice, unless the problem specifically asks you to design a Turing Machine
Terminology for Describing Turing Machines

1. Formal description. Spell out in full the TM’s states, transition functions, etc. Lowest level of description.
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Revisiting Type 0 grammar
Grammars

Definition
A grammar is $G = (V, \Sigma, R, S)$, where
- $V$ is a finite set of variables/non-terminals
- $\Sigma$ is a finite set of terminals
- $S \in V$ is the start symbol
- $R \subseteq (\Sigma \cup V)^* \times (\Sigma \cup V)^*$ is a finite set of rules/productions
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We say $\gamma_1 \alpha \gamma_2 \Rightarrow_G \gamma_1 \beta \gamma_2$ iff $(\alpha \rightarrow \beta) \in R$. 
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We say \( \gamma_1 \alpha \gamma_2 \Rightarrow G \gamma_1 \beta \gamma_2 \) iff \( (\alpha \rightarrow \beta) \in R \). And

\( L(G) = \{ w \in \Sigma^* \mid S \Rightarrow_G^* w \} \)
Example

Consider the grammar \( G \) with \( \Sigma = \{a\} \) with

\[
\begin{align*}
S & \rightarrow \$Ca\# \mid a \mid \epsilon \\
Ca & \rightarrow aaC \\
C\# & \rightarrow D\# \mid E \\
D & \rightarrow \$C \\
E & \rightarrow \epsilon \\
\end{align*}
\]

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\begin{align*}
S & \rightarrow \$Ca\# \Rightarrow \$aaC\# \Rightarrow \$aaE \Rightarrow \$aEa \Rightarrow \$Eaa \Rightarrow aa \\
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& \Rightarrow \$aaCa\# \Rightarrow \$aaaaC\# \Rightarrow \$aaaaE \Rightarrow \$aaaEa \Rightarrow \$aaEaa \\
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The following are derivations in this grammar
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$S \Rightarrow SC \# \Rightarrow Ca \# \Rightarrow aaC \# \Rightarrow aaE \Rightarrow aEa \Rightarrow aEaa \Rightarrow aa$

$L(G) = \{a^i \mid i \text{ is a power of } 2\}$
Grammars for each task

- What is the expressive power of these grammars?
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- Restricting the types of rules, allows one to describe different aspects of natural languages

Noam Chomsky
Grammars for each task

- What is the expressive power of these grammars?
- Restricting the types of rules, allows one to describe different aspects of natural languages
- These grammars form a hierarchy

Noam Chomsky
Type 0 Grammars

Definition
Type 0 grammars are those where the rules are of the form

\[ \alpha \rightarrow \beta \]

where \( \alpha, \beta \in (\Sigma \cup V)^* \)

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$L$ is recursively enumerable (recognizable) iff there is a type 0 grammar $G$ such that $L = L(G)$. 

Expressive Power of Type 0 Grammars
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Thus, type 0 grammars are as powerful as Turing machines.
Recognizing Type 0 languages

Proposition

If $G = (V, \Sigma, R, S)$ is a type 0 grammar then $L(G)$ is recursively enumerable.
Recognizing Type 0 languages

Proposition

If \( G = (V, \Sigma, R, S) \) is a type 0 grammar then \( L(G) \) is recursively enumerable.

Proof.

We will show that \( L(G) \) is recognized by a 2-tape non-deterministic Turing machine \( M \), with tape 1 storing the input \( w \), and tape 2 used to construct a derivation of \( w \) from \( S \).
At any given time tape 2 stores the current string of the derivation; initial tape contains S.

To simulate the next derivation step, M will (nondeterministically) choose a rule to apply, scan from left to right and choose (nondeterministically) a position to apply the rule, replace the substring matching the LHS of the rule with the RHS to get the string at the next step of derivation.

If tape 2 contains only terminal symbols, then M will check to see if it matches tape 1. If so, the input is accepted, else it is rejected. □
Recognizing Type 0 Grammars

Proof (contd).

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- The rules of $S$ will generate an accepting configuration of $M$
- Once (some) initial configuration $q_0 w$ is generated, rules in $G$ will erase symbols to produce the terminal $w$. □