A Simple Laplacian Assignment Model

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1 Intuitions

We have N items and K categories and want to find soft assignments of items to categories given some information, often taking the form of sparse tags or annotations, e.g. for pictures in websites such as Flickr, blog entries, etc.

- **Wisdom of the expert** (information specific for each item irrespectively of other items): item-category similarity values are positive or negative, with the magnitude indicating the degree of association, or zero meaning indifference or ignorance.
- **Wisdom of the crowd** (information about an item in the context of other items): similarity values of a given item to other items, captured with an item-item similarity matrix and its graph Laplacian. We expect similar items to have similar assignments.

2 Laplacian assignments model

- We want to determine soft assignments z_{nk} of each item n to each category k, where $z_{nk} \in [0,1]$, $\sum_{k=1}^K z_{nk} = 1$. Denote $\mathbf{Z} = [z_{nk}]$.
- We are given two sparse similarity matrices: an item-item similarity matrix $\mathbf{W}_{N\times N}$, and an item-category similarity matrix $\mathbf{G}_{N\times K}$.
- We assign items to categories optimally:

$$\min_{\mathbf{Z}} \lambda \operatorname{tr}\left(\mathbf{Z}^T\mathbf{L}\mathbf{Z}\right) - \operatorname{tr}\left(\mathbf{G}^T\mathbf{Z}\right) \quad \text{s.t.} \quad \mathbf{Z}\mathbf{1}_K = \mathbf{1}_N,\ \mathbf{Z} \geq \mathbf{0}$$

  where $\lambda > 0$ and $\mathbf{L}$ is the $N \times N$ graph Laplacian matrix, obtained as $\mathbf{L} = \mathbf{D} - \mathbf{W}$, where $\mathbf{D} = \operatorname{diag}\left(\sum_{n=1}^N w_{nm}\right)$ is the degree matrix.
- This model, called **LASS**, is a quadratic program over NK variables altogether.

3 A simple, efficient algorithm

- We adopt a simple algorithm based on **Alternating Direction Method of Multipliers (ADMM)**, which has simple steps and takes advantage of the structure of the problem.
- Choose a penalty parameter $\rho > 0$ and set

  $$\mathbf{h} = -\tfrac{1}{K}\mathbf{G}\mathbf{1}_K + \tfrac{\rho}{K}\mathbf{1}_N, \quad \mathbf{R}\mathbf{R}^T = 2\lambda\mathbf{L} + \rho\mathbf{I} \text{ (Cholesky decomposition)}$$

  and iterate until convergence:

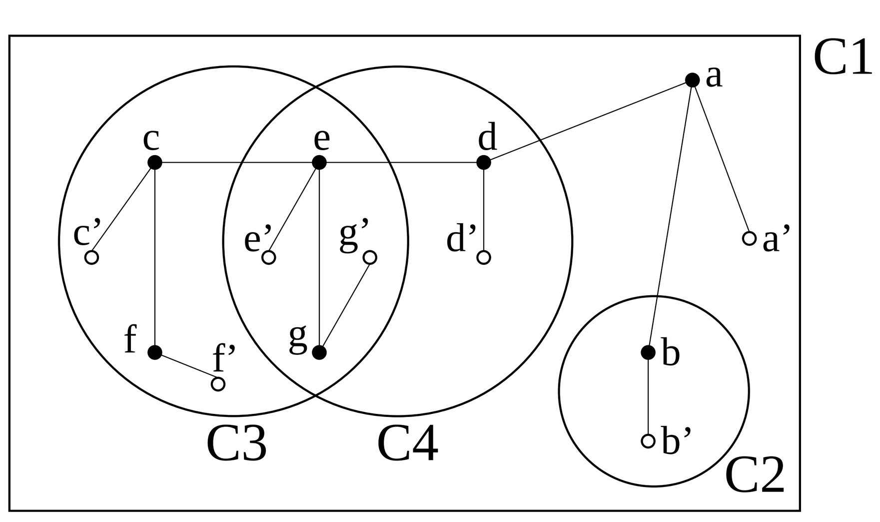
  $$\begin{aligned} &\boldsymbol{\nu} \leftarrow \tfrac{\rho}{K}(\mathbf{Y}-\mathbf{U})\mathbf{1}_K - \mathbf{h}, && \mathbf{Z} \leftarrow (2\lambda\mathbf{L}+\rho\mathbf{I})^{-1}(\rho(\mathbf{Y}-\mathbf{U}) + \mathbf{G} - \boldsymbol{\nu}\mathbf{1}_K^T) \\ &\mathbf{Y} \leftarrow (\mathbf{Z}+\mathbf{U})_+, && \mathbf{U} \leftarrow \mathbf{U} + \mathbf{Z} - \mathbf{Y} \end{aligned}$$

  where $\mathbf{Z}_{N\times K}$ are the primal variables, $\mathbf{Y}_{N\times K}$ the auxiliary variables, $\mathbf{U}_{N\times K}$ the Lagrange multiplier for $\mathbf{Y} = \mathbf{Z}$, and $\boldsymbol{\nu}_{N\times 1}$ the Lagrange multipliers for $\mathbf{Z}\mathbf{1}_K = \mathbf{1}_N$.

4 Out-of-sample mapping

- Given a new, test item $\mathbf{x}$, along with its item-item and item-category similarities $\mathbf{w} = (w_n)$, $n = 1,\dots,N$ and $\mathbf{g} = (g_k)$, $k = 1,\dots,K$, respectively.
- Out-of-sample assignment $\mathbf{z}$ for $\mathbf{x}$ is the Euclidean projection $\Pi(\bar{\mathbf{z}} + \gamma\mathbf{g})$ of the K-dimensional vector $\bar{\mathbf{z}} + \gamma\mathbf{g}$ onto the probability simplex, where $\gamma = 1/2\lambda(\mathbf{1}_N^T\mathbf{w}) = 1/2\lambda\sum_{n=1}^N w_n$ and $\bar{\mathbf{z}} = \frac{\mathbf{Z}^T\mathbf{w}}{\mathbf{1}_N^T\mathbf{w}} = \sum_{n=1}^N \frac{w_n}{\sum_{n'=1}^N w_{n'}}\mathbf{z}_n$ is a weighted average of the training points' assignments, and so $\bar{\mathbf{z}} + \gamma\mathbf{g}$ is itself an average between this and the item-category affinities.
- This mapping as a function of λ **represents the tradeoff between the crowd ($\mathbf{w}$) and expert ($\mathbf{g}$) wisdoms**. It is quite different from the simple average of $\bar{\mathbf{z}}$ and $\mathbf{g}$ and may produce exact 0s or 1s for some entries.

5 Experimental results



data	GT	Pos	Neg
a	1	1	not 2
b	1,2	2	
c	1,3	1	
d	1,4	4	not 3
e	1,3,4	1,3	
f	1,3	3	
g	1,3,4	3,4	

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	C1	C2	C3	C4
a'	1	0	0	0
b'	0	1	0	0
c'	1	0	0	0
d'	0	0	0	1
e'	0.5	0	0.5	0
f'	0	0	1	0
g'	0	0	0.5	0.5

LASS (pos. only)

	C1	C2	C3	C4
a'	0.57	0.28	0.10	0.05
b'	0.40	0.60	0	0
c'	0.30	0	0.70	0
d'	0.36	0.08	0.33	0.23
e'	0.29	0	0.68	0.03
f'	0.05	0	0.95	0
g'	0	0	0.82	0.18

LASS (pos. & neg.)

	C1	C2	C3	C4
a'	0.90	0	0	0.10
b'	0.70	0.30	0	0
c'	0.60	0	0.40	0
d'	0.70	0	0	0.30
e'	0.57	0	0.37	0.07
f'	0.35	0	0.65	0
g'	0.24	0	0.53	0.23

GT: fence tree white building grass green house man sky
Neg: black girl hair woman yellow
Pred: tree white fence green sky house mountain man people

GT: black tree white flower man painting red woman yellow
Neg: blue girl green picture water
Pred: black tree white man woman eye hair red hat

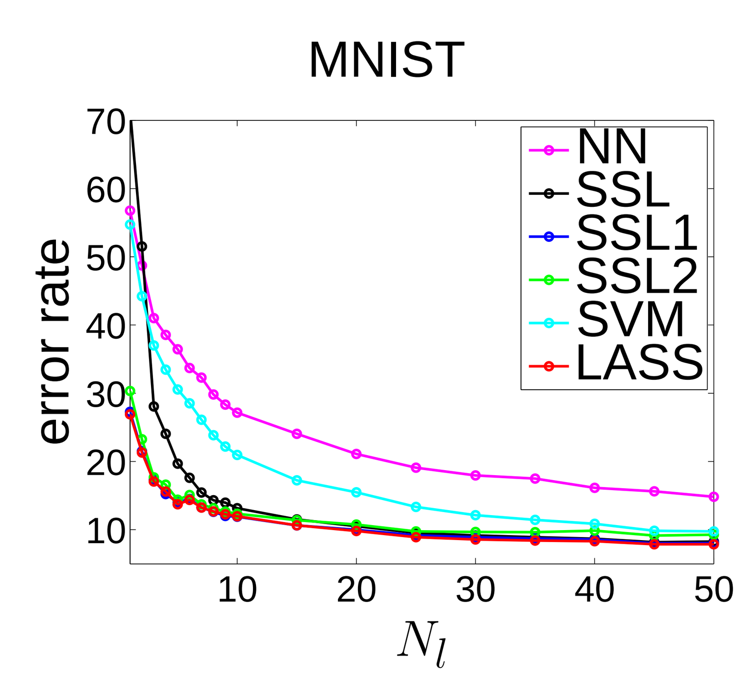
GT: dog show tree black fence grass green man shirt
Neg: old people red smile white
Pred: tree green dog grass show man house sky jump

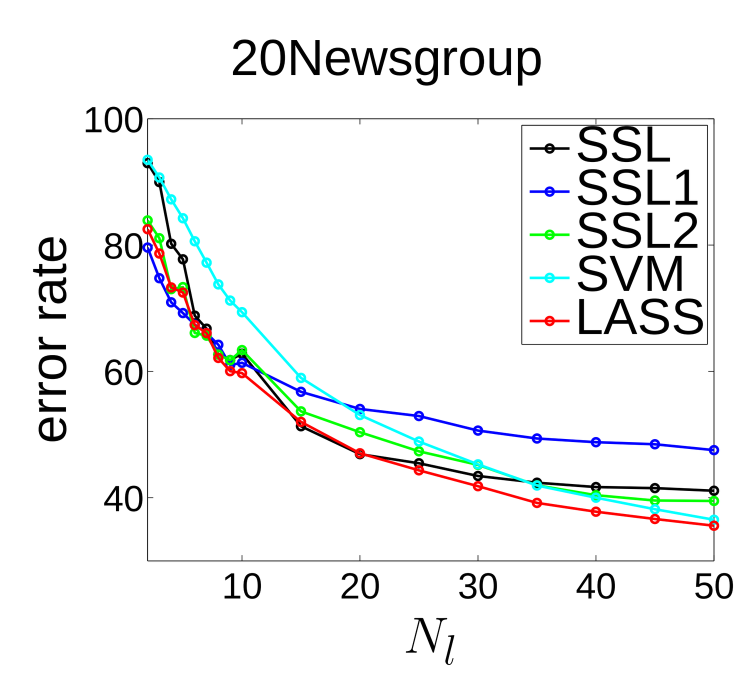
GT: money old silver coin man woman
Neg: face girl green hair smile
Pred: money old black silver coin man

GT: boat green river blue red tree water
Neg: brown grass man woman yellow
Pred: green river boat tree white red water

GT: bird stone water black legs rock
Neg: brown green old sky tree
Pred: bird stone water rock beak black

ESP Game subset. Green: positive affinity. Red: negative affinity. Cyan: true positive in predictions.





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