Topological Entropy in the Hénon map

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Abstract

Compared to other areas of physics and math, chaos theory is a relatively new field that has many new areas to be explored. Although chaotic systems are everywhere in nature, many real systems are too complex to be modeled on regular computers. The Hénon map is a simple iterated map that displays chaotic behavior in two dimensions that is easily computed. Topological entropy is a measure of the complexity of a system that can be used to compare different configurations of the Hénon map. Changes in topological entropy over varying k values in the Hénon map may lead to clues at how changes in k affect the dynamics of the system. Generalizing the effects of changing k may be possible, giving us new theories about chaotic systems that can be tested on more realistic systems.

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Chapter 1

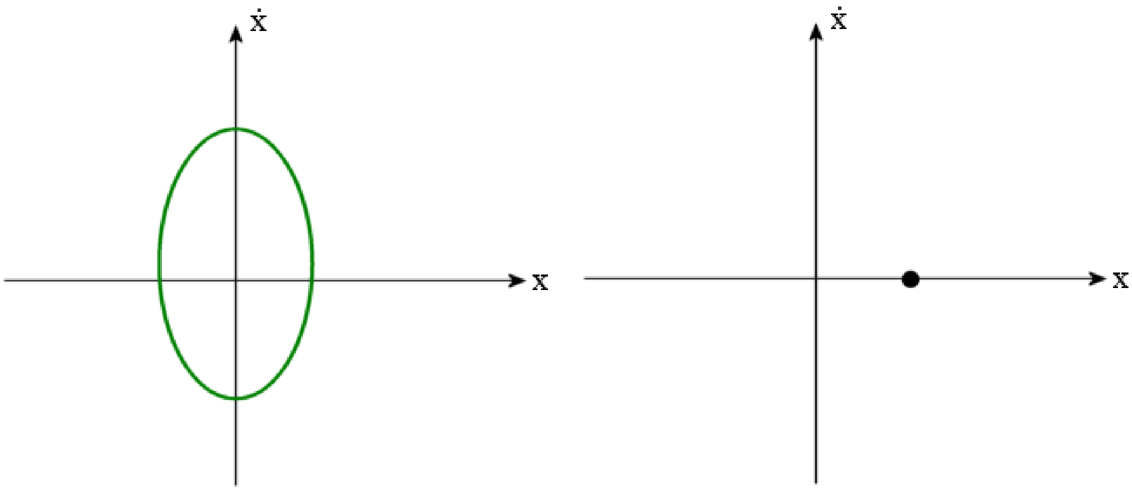
Background

1.1 Introduction

Because many real-world phenomena are very complex, they are often distilled into simpler models. By necessity, these models ignore many variables that may be present, but still attempt to capture the essential behavior of a system. Although linear models are easily solved, it is only recently that physicists, mathematicians and other scientists have been able to look at the behavior of nonlinear models like the Hénon map in detail, due to the availability of increasingly powerful computers. However, many of the tools used to study these models were developed over the past century and before.

1.2 History of Nonlinear Dynamics and Chaos

One of the first tools still used today in the study of nonlinear dynamics is the surface of section or Poincaré section, named after Henri Poincaré. Poincaré won a mathematical contest to come up with a solution to the three-body problem, though he did not actually solve the problem. In fact, Poincaré showed that the three-body problem is impossible to solve analytically[1]. However, in the process of looking for a solution, he developed a simplified way of visualizing very complex behavior of the resulting trajectories. Instead of plotting the entire trajectory, Poincare focused on the particles position and momentum at discrete-time intervals[1]. The simple example of an undamped pendulum, with the equation of motion $\ddot{\theta} = -\omega^2\theta$, allows for easy visualization of the Poincaré section[2-3]. The phase plot of an undamped pendulum is an ellipse, while the Poincaré section is simply a point (figure 1).



(a) (b)

figure 1 Phase plot (a) and Poincaré section (b) of a undamped pendulum

This is due to the fact that the Poincaré section is created by plotting the phase space position only once per period[1]. In more complex systems, which may or may not be periodic, multiple points will be plotted, but this is still a much reduced version that allows for easy visualization for sometimes very complex behavior[1]. Others, such as Lyapunov, Feigenbaum and Hénon, also contributed to the field of nonlinear dynamics and chaos during the twentieth century and beyond. Additionally, faster computers and programs such as Mathematica and MatLab have allowed the study of chaotic systems to be studied much more easily than in the past[3]. This has caused a boom nonlinear dynamics and chaos research that still continues to today, and as a result, chaos theory has been applied to many different situations.

1.3 Applications

The methods developed by Poincaré and others are used to study chaotic behavior in a wide variety of systems including models of the behavior of the heart, changes in biological populations due to predator-prey interaction, fluctuations in financial markets[2-3], and the ionization of molecules under the influence of a magnetic field [4]. Chaos can occur in all of these systems because it only requires two main properties: sensitivity to initial conditions and sensitivity to a parameter. Neither high dimension nor complex equations are needed for chaos. Many one-dimensional systems, such as the logistic map, behave chaotically[3].

One of the most concrete examples of a chaotic system is the damped, driven pendulum, which has the equation of motion

$$\ddot{\theta} + \gamma\dot{\theta} + sin\,\theta = F\,cos\,w_d t$$

where F is the strength of the driving force and w_d is its angular frequency[2]. Although this system is often solved by simplifying sin θ to θ, in reality it is a nonlinear system. This system exhibits both properties of a chaotic system. Small changes in the initial conditions of $\ddot{\theta}$, $\dot{\theta}$, and θ can result in vastly different behavior, and small changes in its parameter, F, will do the same.

There are no analytic solutions to the above equations, however, so they must be solved numerically. To do this, they must be rewritten in terms of two, first-order equations:

$$\dot{\omega} = -\gamma\omega - sin\,\theta + F\,cos\,w_d t\,, \qquad \dot{\theta} = \omega$$

The solutions can then be plotted by a program, either in real space or in phase space. Chaotic systems can give rise to varying phenomena, such as chaotic attractors and tangles[2-3]. There are two types of tangles: homoclinic and heteroclinic[5]. Because the area preserving Hénon map creates a homoclinic tangle, I will be focusing on this type.

1.4 Homoclinic Tangles

Homoclinic tangles are complex objects, so to understand what they are, it is first helpful to understand their components. Homoclinic tangles are made up of stable and unstable manifolds attached to a fixed point[5]. In one dimension, a fixed point can be stable if nearby points move towards it, or unstable if nearby points fly away from it. A stable fixed point occurs at the minimum of a damped harmonic oscillator—a pendulum will always come to rest straight down if no forces are applied. An unstable fixed point can be visualized as a ball at the top of a hill: small perturbations away from the fixed point will result in the ball falling off (figure 2a). In two-dimensions, fixed points are a little more complicated. Stable fixed points must be stable in all directions, which is basically the same idea as in one dimension. However, unstable fixed points occur in two varieties.

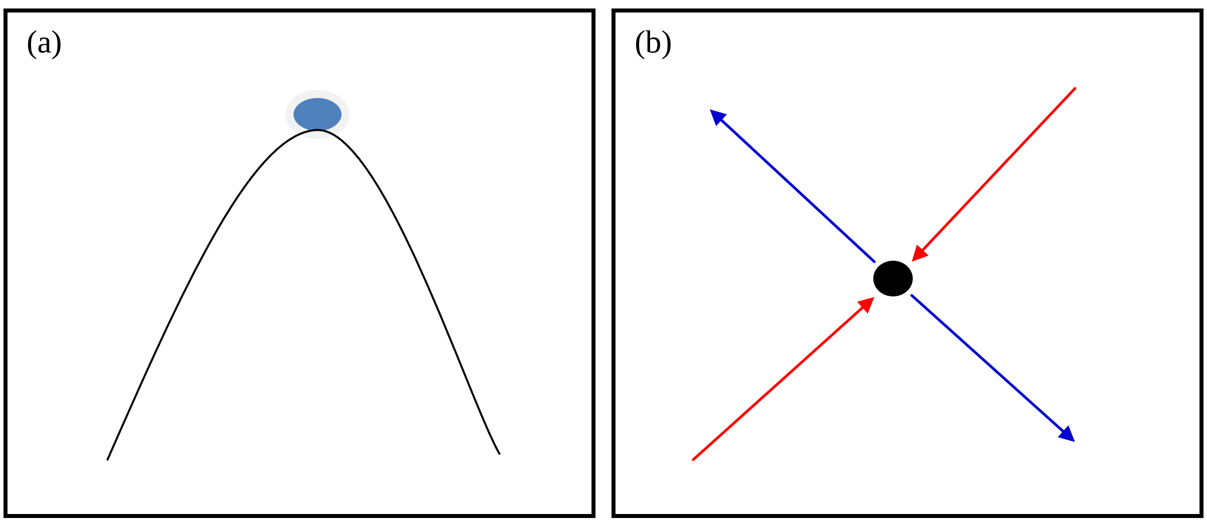


figure 2 (a) Small perturbations will result in large movement away from an unstable fixed point, such as a ball on top of a hill. (b) A hyperbolic fixed point with stable (red) and unstable (blue) directions.

First, an unstable fixed point may be unstable in all directions, similar to the one dimensional case. Second, unstable fixed points may be unstable in all directions, or maybe unstable in one directions but not another. This is known as a hyperbolic fixed point (figure 2b). This type of fixed point is the type found on homoclinic tangles.

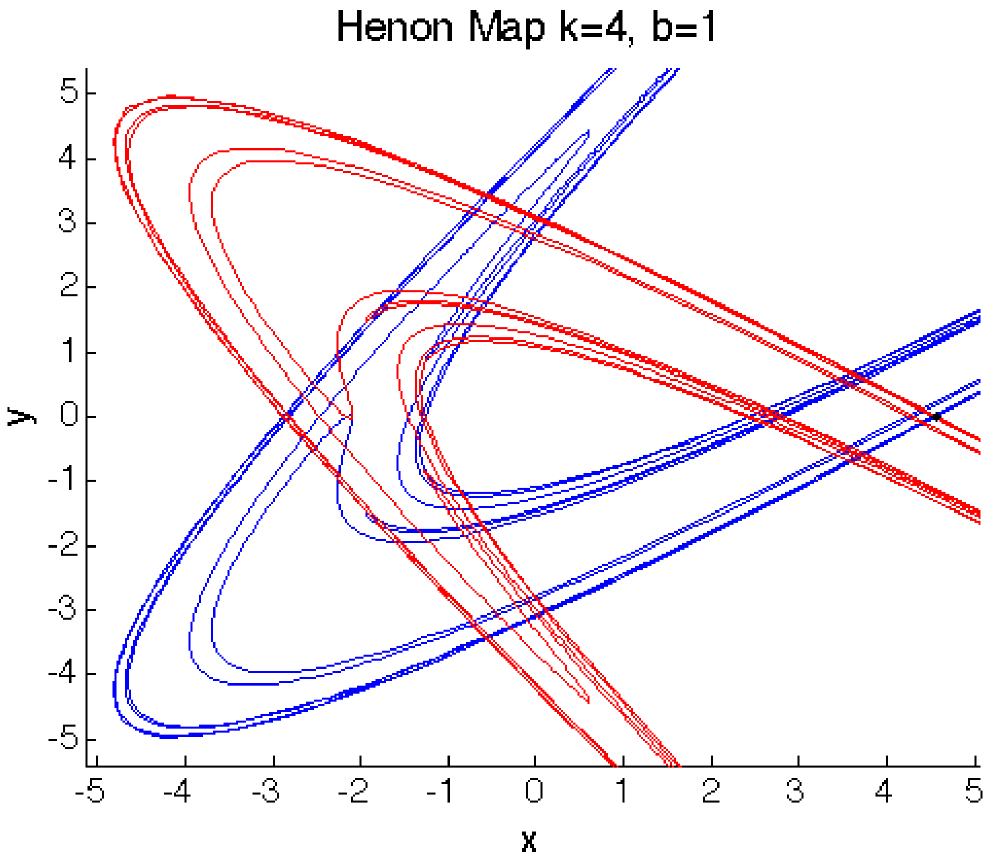


figure 3 Stable and unstable manifolds of the Hénon map after five iterates

The second important component of homoclinic tangles are manifolds. There are two types of manifolds, stable and unstable, attached to a hyperbolic fixed point (figure 3). They can be thought of as extending the trajectories of the stable and unstable directions attached to a hyperbolic fixed point. Although they do not remain straight lines, a point on the stable manifold will still move towards the fixed point, while a point on the unstable manifold will move away. A manifold can never cross itself, which results in their winding behavior, but unstable and stable manifolds will intersect an infinite number of times. This complicated bending and twisting gives rise to the name "tangle."

Homoclinic tangles consist of infinitely long portions of stable and unstable manifolds attached to a single fixed point. For clarity, only a short portion of the stable manifold is usually pictured, while the unstable manifold is computed for longer lengths (figure 4**)**. Information about the dynamics of the system can be determined by properties of the tangles, and more information is acquired as the system is iterated further. There is another type of tangle known as the heteroclinic tangle, which involves the manifolds of two fixed points interacting, but this type will not be covered.

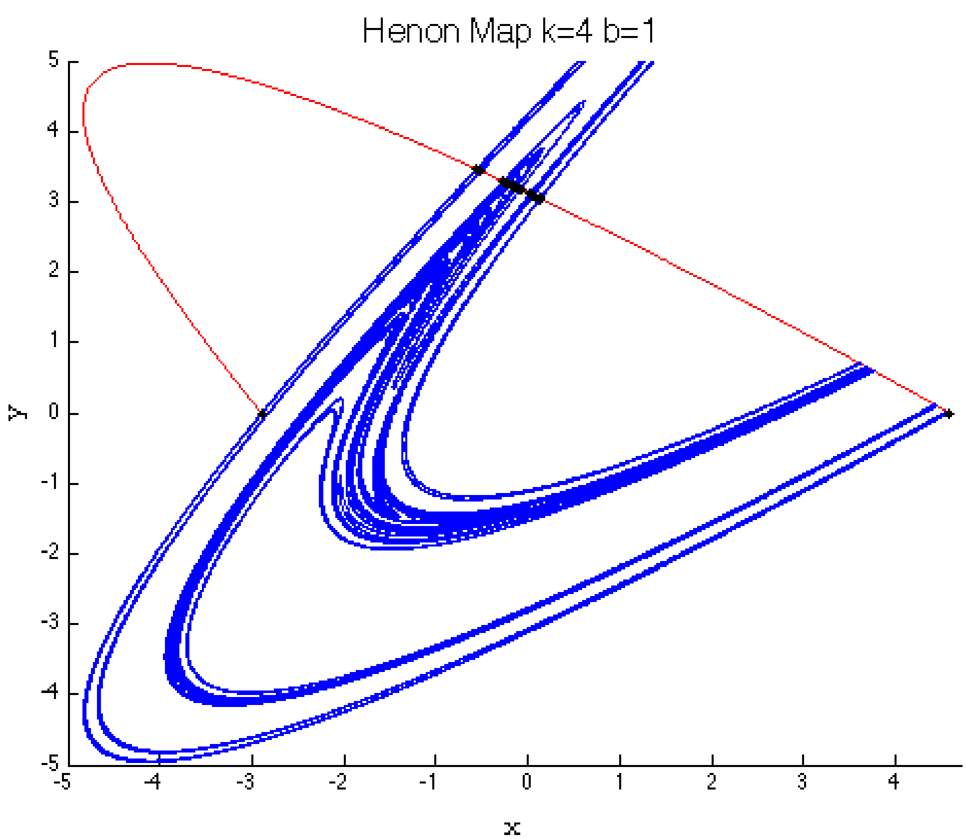


figure 4 The Hénon map after many iterates

1.5 The Hénon map

One of the simplest two-dimensional chaotic systems is the Hénon map, which is given by the two equations

$$x_{n+1} = y_n - k + x_n^2$$

$$y_{n+1} = -bx_n$$

Different from the damped, driven pendulum example, which is continuous in time, the Hénon map is an iterated map, or discrete in time, that maps the (x,y) plane onto itself and exhibits chaotic behavior[2-3]. It was created by Michel Hénon as a simplification of the Poincaré section of the Lorenz system [2] and has since then been studied a great deal because of its ease of computation and interesting behavior.

The canonical version of the Hénon map has $b < 1$, so the area shrinks after each iterate. It also has a fixed k value. The most interesting property of the canonical Hénon map is its chaotic attractor[2-3]. Similar to a fixed point, points nearby the attractor will move towards it and eventually follow its path. However, a chaotic attractor is not made up of one point, but is instead a complex pattern. This version of the Hénon map has been very well studied, but there are many other types. In addition, there are many other two-dimensional maps that exhibit chaotic behavior, such as the standard map and the Baker map [2].

1.6 Topological Entropy

One good way to measure the complexity of a homoclinic tangle is to look at its topological entropy: the more complex a system is, the more topological entropy it will have. The topological entropy of a system is related the growth rate of a material line. In chaotic systems, this is usually an exponentially increasing growth rate. When looking at iterated maps like the Hénon map, an estimate of the lower bound of the topological entropy can be made after each iterate, which will eventually converge to the true topological entropy after infinite iterations. Because of this, the lower bound must monotonically increase: more information can never result in less complexity. At best, the complexity will remain the same. Often, the lower bound begins at zero because there is no chaotic behavior and the stable and unstable manifolds do not intersect except at the fixed point. The lower bound estimate increases each time the stable and unstable manifolds cross at higher iterates. In figure 5, this occurs at iterates three, five and twelve, and the entropy is converging to a value of about $h = 0.6$.

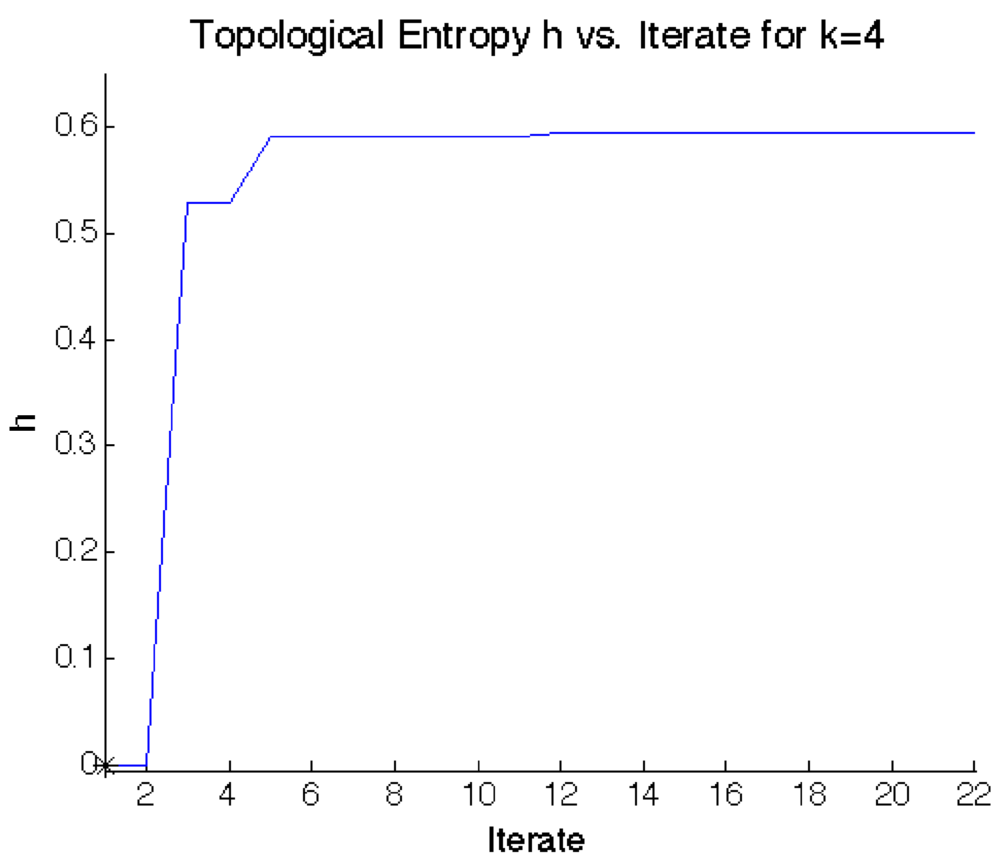


figure 5 Topological entropy of the Hénon map at k = 4 for 22 iterates. The lower bound for the first two iterates is zero, then rapidly increases.

Because topological entropy is related to complexity, it can be used to compare maps at different parameter values, which can have drastically different behavior even when the change in parameter is small due to their chaotic nature. The appearance of new fixed points and periodic orbits at certain parameter values can make the system more complex, but sometimes their effects are not immediately obvious. The topological entropy may give hints at when a periodic orbit starts to affect the system, and may allow for a way to visualize the changing complexity of the system as a whole as parameters are varied.

Chapter 2

Topological Entropy Over a Range of *k* Values

2.1 The Hénon Map Revisited

As explained previously, the Hénon map is a well-studied map that exhibits chaotic behavior with the simple set of equations:

$$x_{n+1} = y_n - k + x_n^2$$

$$y_{n+1} = -bx_n$$

where b and k are the parameters of the map. For the purposes of our simulations, we set $b = 1$ and varied the k values from $k = 2$ to $k = 4.5$ in increments of 0.01. When $b = 1$, the Hénon map is area-preserving. This means that the map neither stretches or shrinks after each iterate, and the manifolds behave somewhat like trajectories in real space. When $b = 1$ the Hénon map is also invertible, allowing for the calculation of the stable manifold by inverting the Hénon map:

$$x_{n-1} = -\frac{1}{b}y_n$$

$$y_{n-1} = x_n + k - (\frac{1}{b}y_n)^2$$

When $b = 1$, the Hénon map can be thought of as a simple model for ionization: chaotic behavior occurs inside the tangle, while a particle escapes when it crosses the stable manifold.

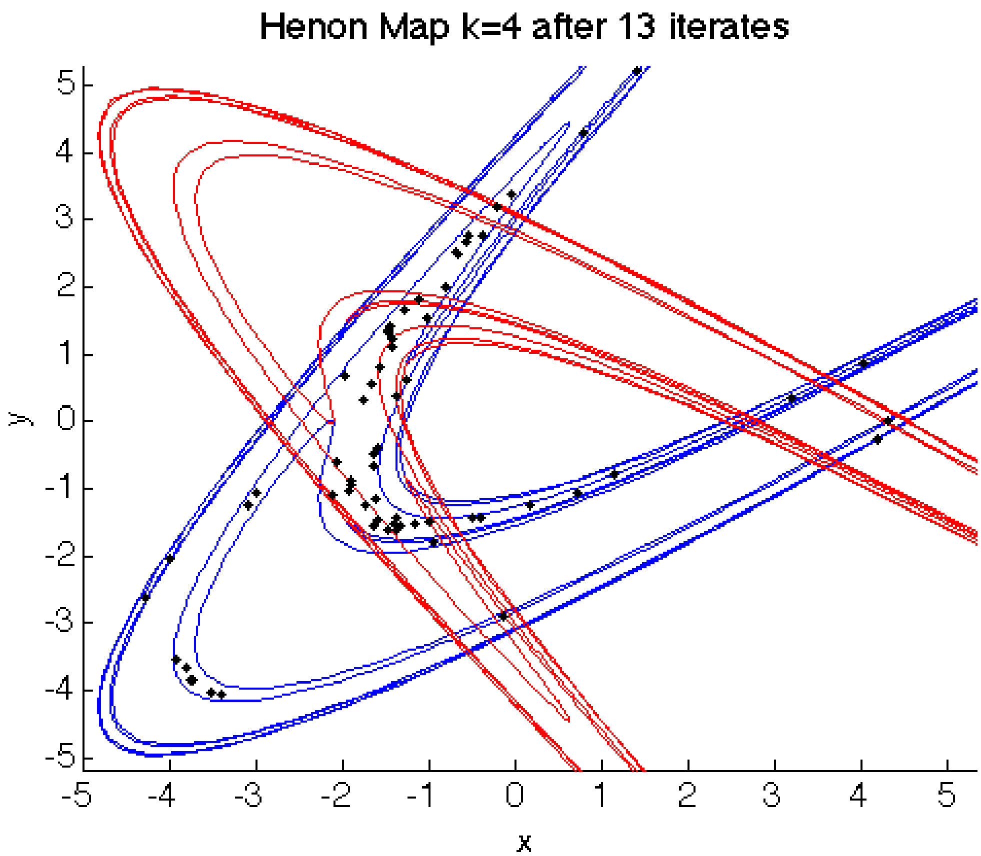


Figure 6 Black points are the remaining points inside the Hénon map after 13 iterates. Those above the stable manifold have escaped on this iterate.

In order to see how the appearance of a new period two fixed point affects the Hénon map's dynamics, we compared topological entropy over a range of *k* values. First, we looked a shorter *k* range of 3.4 to 4.0 because the period two orbit that appears at *k* = 3 seems to get large enough to start affecting the map in this range. However, it was difficult to tell how the dynamics was affected without anything to compare it to, so we extended the *k* range to 2 to 4.5 to overall behavior, using increments of 0.01 between *k*. After each iterate, the topological entropy and other information was extracted. As *k* increases, the Hénon map becomes more complex at earlier iterates, which take longer to compute. Because of this, an area cutoff was added which stopped the program after the area between two segments of unstable manifold become too close together. This caused the results for different *k* values to run for different amount of iterates. In addition, intersection errors in the unstable manifold sometimes caused the program to stop early, but each *k* value yielded at least ten iterates worth of data for comparison.

2.2 Single *k* Entropy Results

Plotting entropy for a single *k* value results in a graph such as figure 7. The first lower bound of entropy is always zero because there is essentially no complexity to the system. The second value is also always zero for our range of *k* because it does not increase complexity enough to affect anything. After the third iterate, the lower bound is increased to around 0.53 for all *k* in this range. However, after the third iterate, the behavior of the topological entropy becomes much more unpredictable. In addition, there

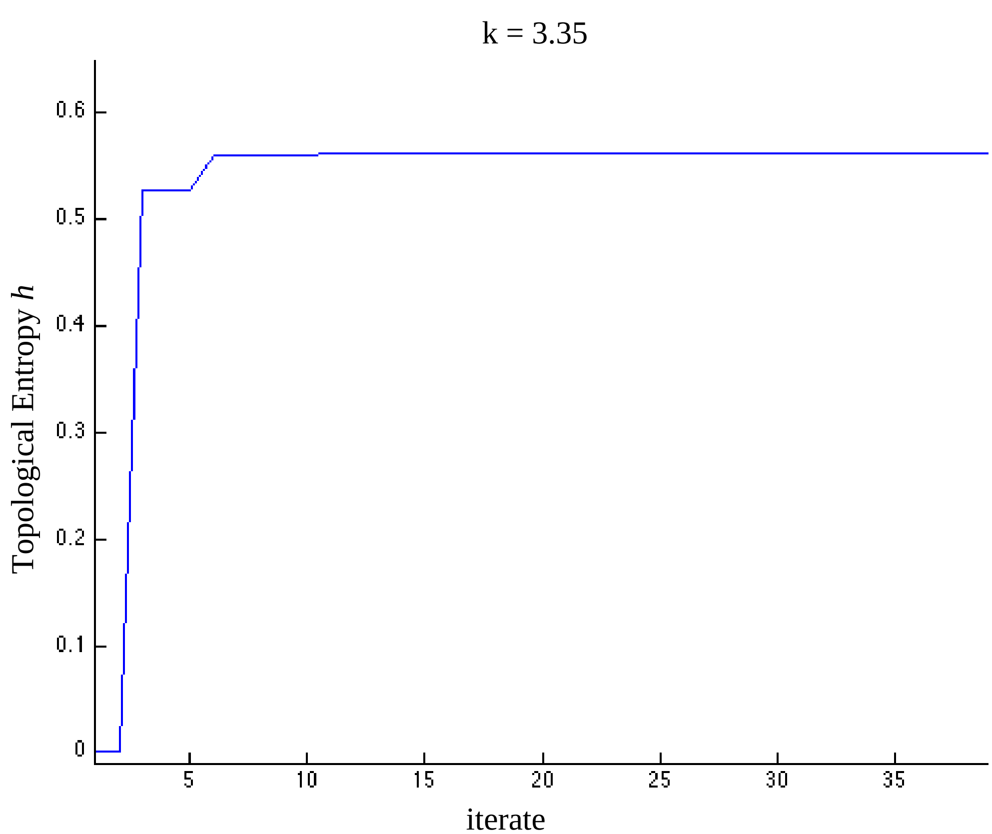


Figure 7 topological entropy vs. iterate for k=3.35

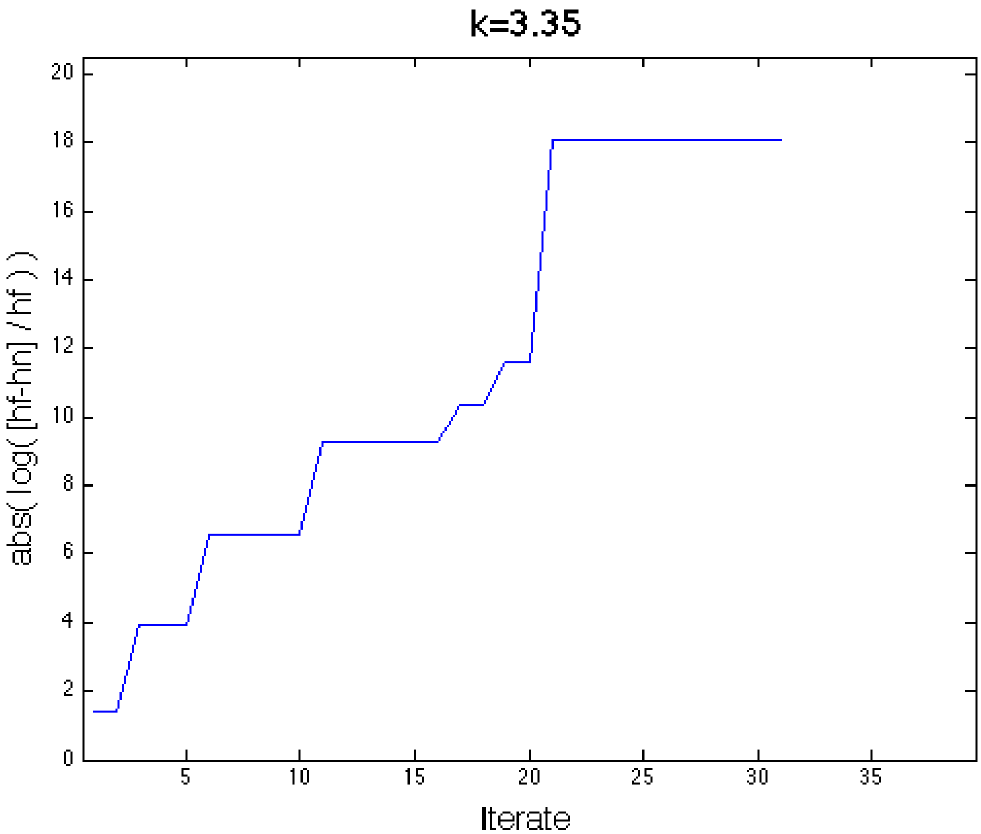


Figure 8 Using normalized entropy differences highlights small changes in *h*

may be small changes in entropy that are difficult to see by inspecting the entropy plots.

In order to see smaller changes in entropy more easily, a normalized entropy difference can be used. In this case, we used the equation

$$\left|\ln[(e^{h_f} - e^{h_n})/e^{h_f}]\right|$$

to highlight small changes in entropy as in figure 8. This reveals that for some *k* values there are many, very small changes in entropy between iterates 10 to 25 not seen in the original entropy plot. The ideal way to graph over large *k* ranges would show how these small changes in entropy are affected by changing *k* values.

2.3 Plotting Entropy Results Over a large *k* range

In order to compare entropy values over a large *k* range, we first used normalized entropy differences for the false color graph as a way to highlight entropy changes over this range. If there were no crashes in the MatLab code for a particular *k* value, the last entropy difference was set to 20 instead of leaving these values at infinity from the natural logarithm. If the code crashed part way through, we set the entropy difference to infinity (white) for all values after that point, and if there was an erroneous intersection we set the differences to negative infinity (black) for all iterates where this occurred.

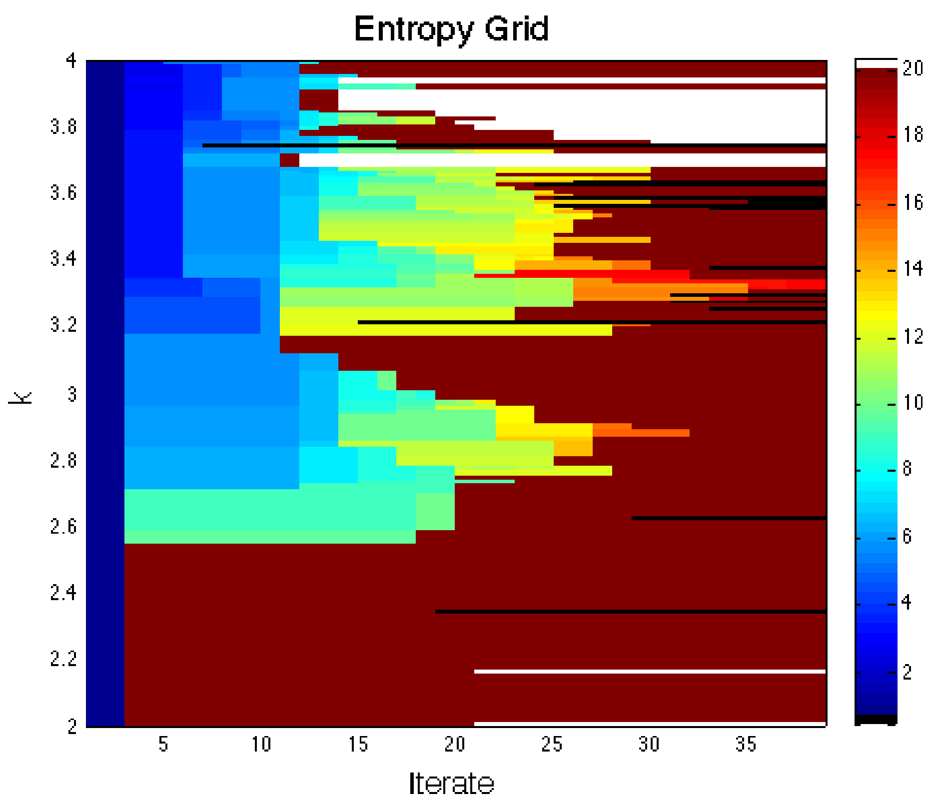


figure 9 False cover plot from k = 2 to k = 4

The main advantage of viewing the entropy in this way is the ability to clearly distinguish the changes in entropy for each *k* value. However, because the differences are normalized compared with each *k* value's final entropy, it is difficult to compare changes between *k* values. Although some general trends are noticeable, such as the relatively steady entropy trend at low *k*, other trends are not. In addition, regions of nonmonotonicity are usually not noticeable in this graph, which makes it difficult to discern where the code has errors. This is especially difficult to see when looking over a large range of *k* values.

Although the previous graph allowed us to see generally where entropy changed, we needed to find a better way to see where larger jumps in entropy occurred. In order to do this, we plotted e^h for each *k* value on the same graph. Setting low *k* to green and high *k* to blue, we can see that lower bound estimates jump to higher values at earlier iterates as *k* increases. Details are difficult to make out in some sections where entropy is

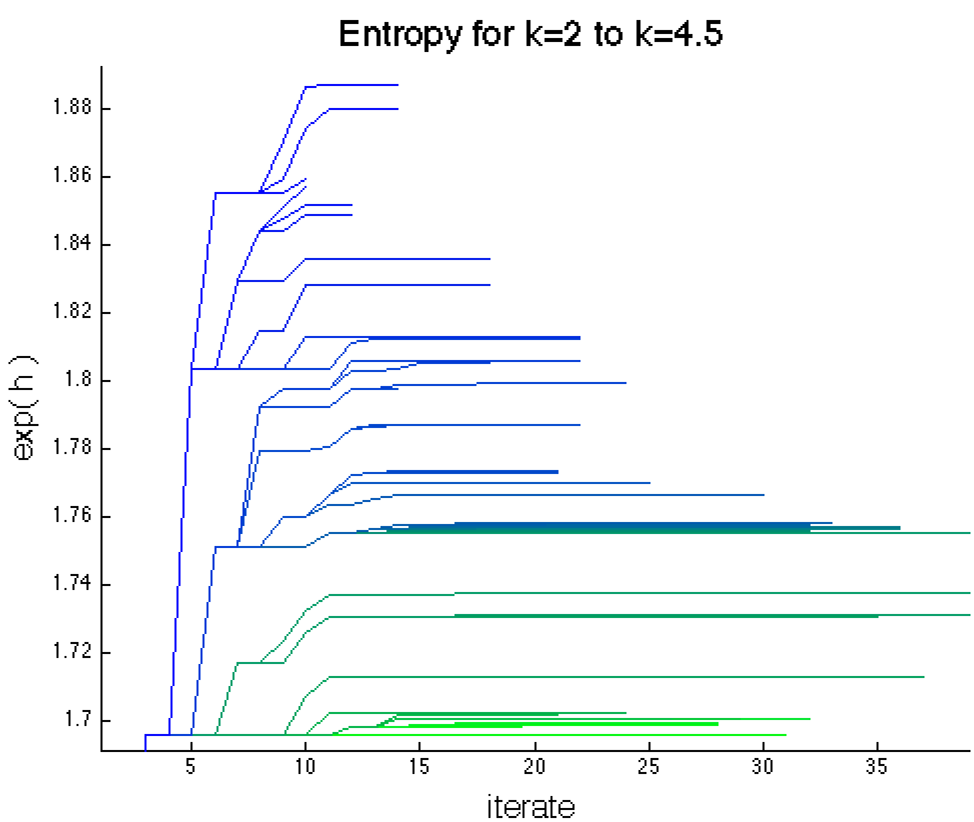


Figure 10 All entropy plots for k = 2 to k = 4.5 in increments of 0.01 Green is low k while blue is high k

changing slightly at each *k*, but it is easier to figure out where these sections are and take a closer view of them with this method.

2.4 Entropy Groups

There are six main groups of data revealed by plotting entropy in this way. The first group is from k = 2.0 to 2.54 (figure 11). This group has the same entropy profile for every iterate and is represented by a single line at the lowest portion of the graph. The second group goes from k = 2.55 to 3.17 (figure 12). This is a group where jumps in entropy start to occur. At the lower end of this range, the jumps occur at late iterates, but as k increases, the jumps begin to occur earlier. A period two orbit appears at k = 3, which could account for some of this complexity, but we believe that it is too small to affect anything at this point because it lies inside a much larger period three tangle. This area is considered to end when the first jump in entropy is much greater than previous iterates. The third group therefore occurs at k = 3.18 to 3.34 (figure 13). Although this is a somewhat sizeable range of *k* values, this area results in only four different entropy profiles.

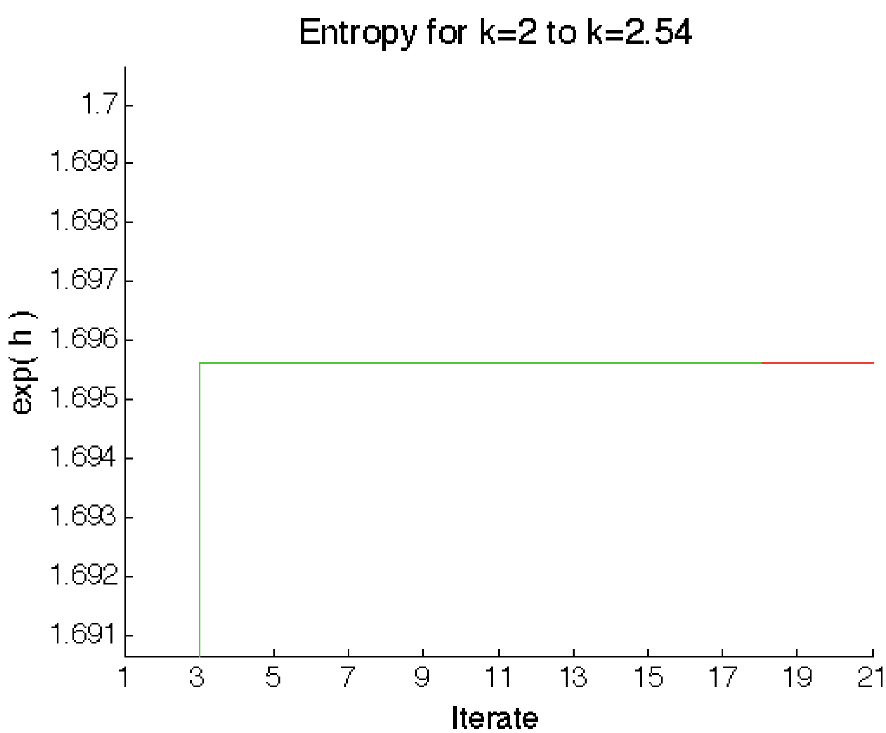


Figure 11 First entropy group

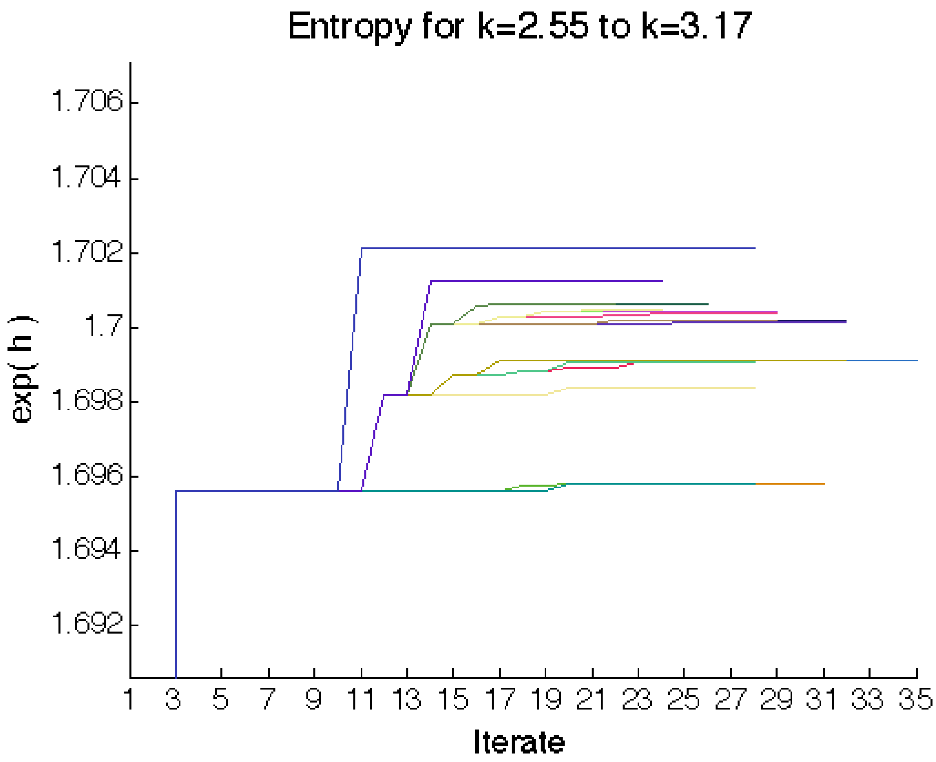


Figure 12 Second entropy group

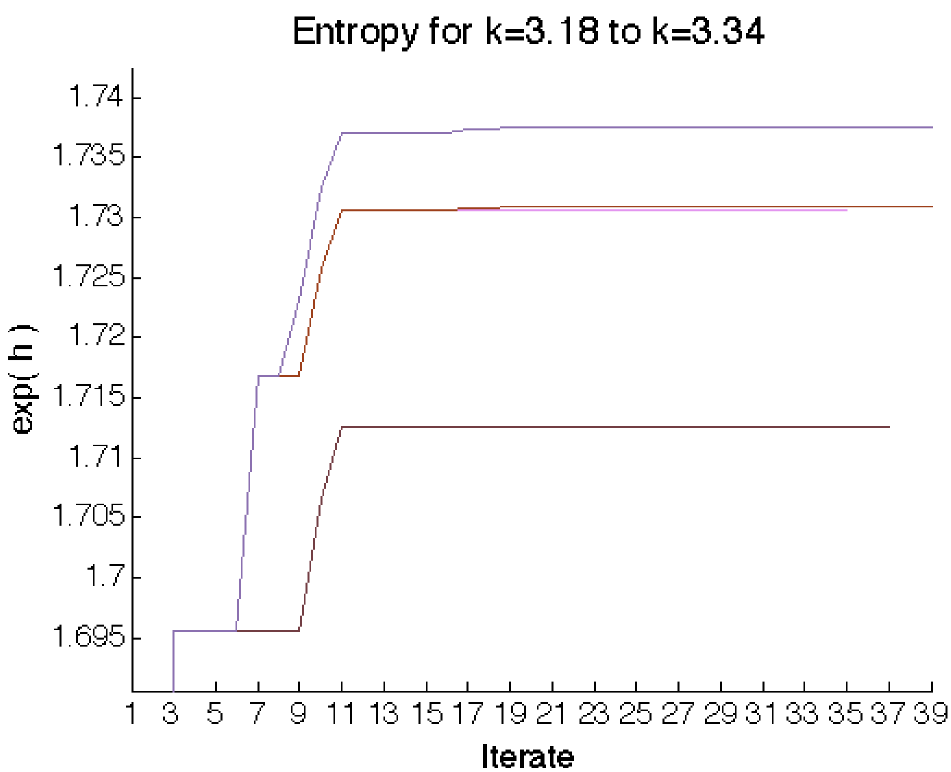


Figure 13 Third entropy group

The fourth group occurs between k = 3.35 to 3.74. This is the second main area when complexity starts to creep in from high iterates. It is difficult to see on the plot over all *k* values, so it is helpful to look at it on its own. This also shows that entropy is self-similar because this view looks much the same as the total view. This group also shares another feature, which is that the second jump in entropy occurs at the eleventh iterate. Once again, as k increases the jumps in entropy start occurring at earlier and earlier iterates.

(a) (b)

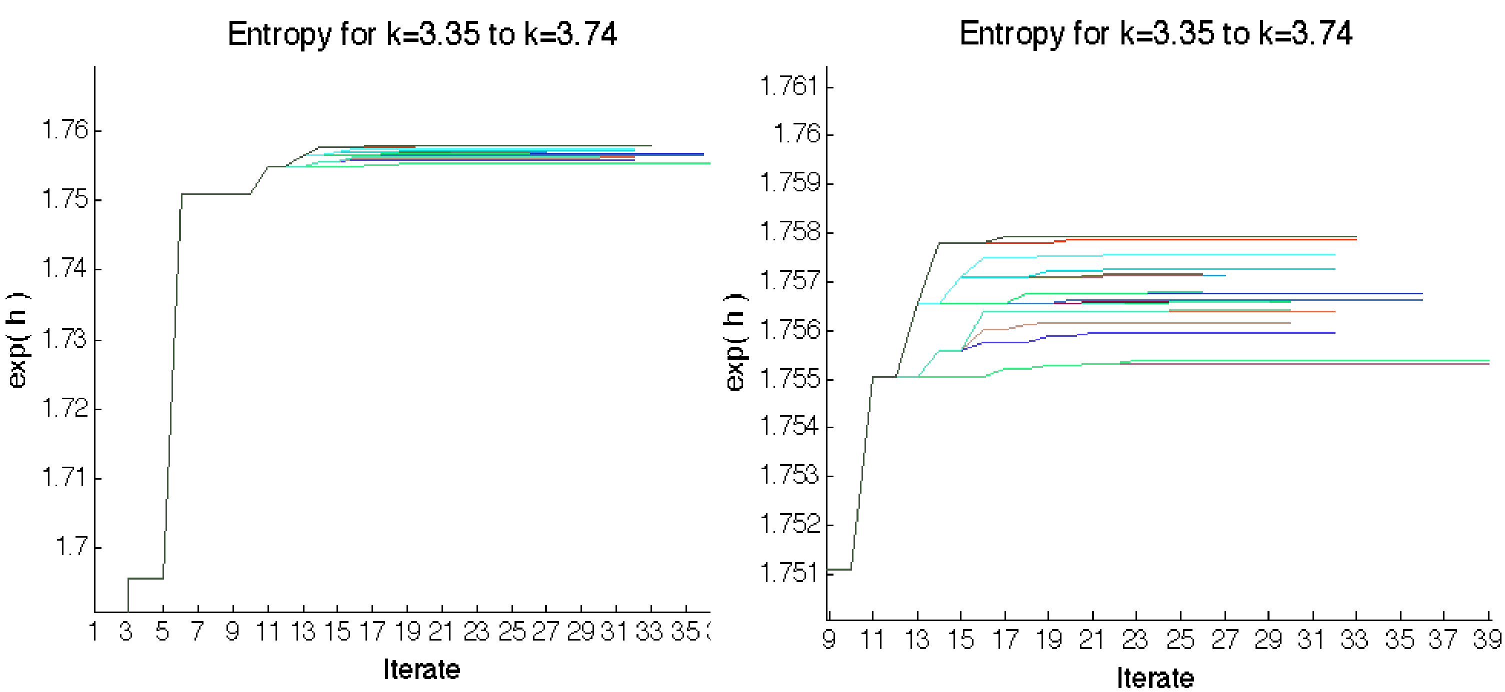


Figure 14 Fourth entropy group (a) full view and (b) zoomed view.

The next two sections are areas of relative little activity. The fifth consists of k = 3.74 to 3.98, all of which have a second jump in entropy at the seventh iterate. The final group occurs from k = 3.99 to 4.5, where the second jump in entropy occurs at iterate five. Although the second jumps occur on different iterates, otherwise these two sections behave relatively the same. In addition, many of the runs in the last section ended early, so there may be more information that isn't pictured in this graph that appears at late times.

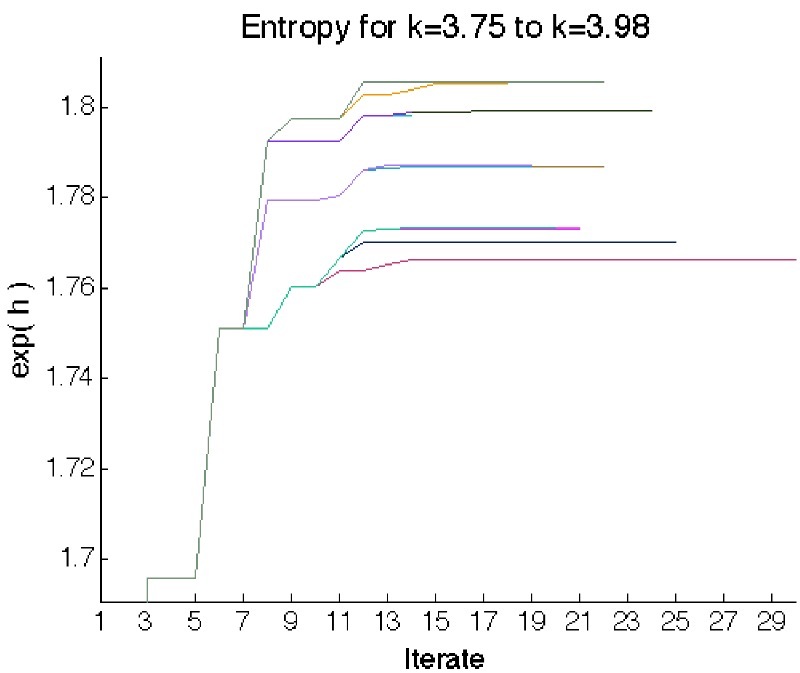


Figure 15 Fifth entropy group

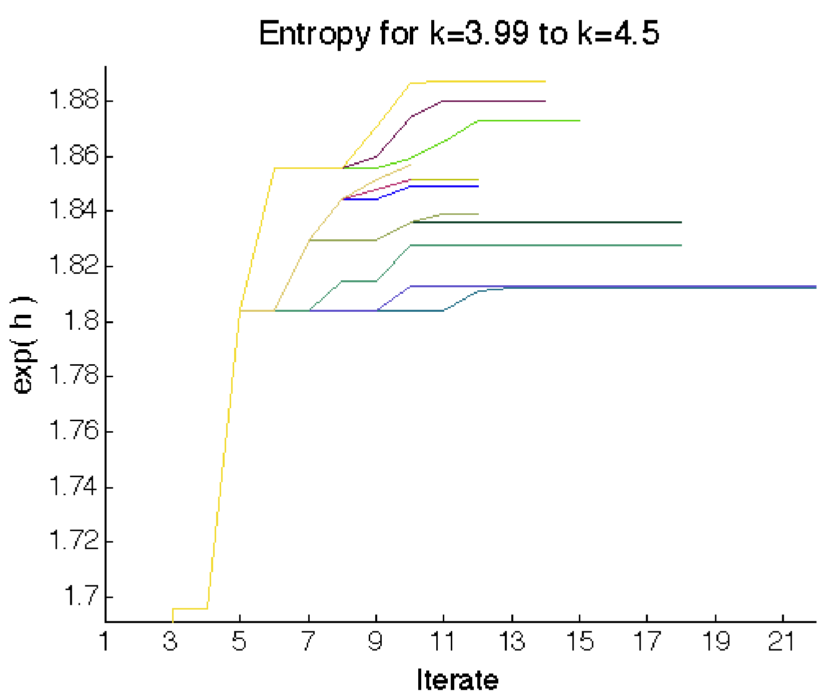


Figure 16 Sixth entropy group

These changes in how entropy jumps at different iterates can be viewed as "entropy fronts" coming in from late times to earlier times. For example, as the second group transitions into the first group, all the complexity that appeared in small jumps now appears in one large jump at k = 3.18, and then remains relatively flat. Another place to easily picture an entropy front coming in is between the fifth and sixth groups. In the fifth group, all *k* values jump up at the fifth iterate, whereas in the sixth group all *k* values jump immediately up to an even higher value at the fourth iterate.

These entropy fronts are useful for looking at where changes in *k* result in large jumps in entropy. In addition, there are times one *k* value has many small changes at high iterates, while the next *k* value only 0.01 away may only have one, larger jump in entropy for the same iterate interval. One reason this occurs is that the Hénon map become larger as *k* increases, so that more information is revealed on each iterate. However, this alone does not explain how the small changes occur.

Chapter 3

Conclusions

3.1 Entropy Fronts

We have determined a good way to view changes in entropy over a large range of *k* values. Although sometimes this requires extra plotting to see areas with many small changes in entropy, it is easy to focus on these areas and plotting them is simple. In addition, this method of plotting entropy has been useful in determining when large jumps in entropy occur between small changes in k. In addition, unlike our previous method of plotting entropy, this method highlights the fractal nature of changes in

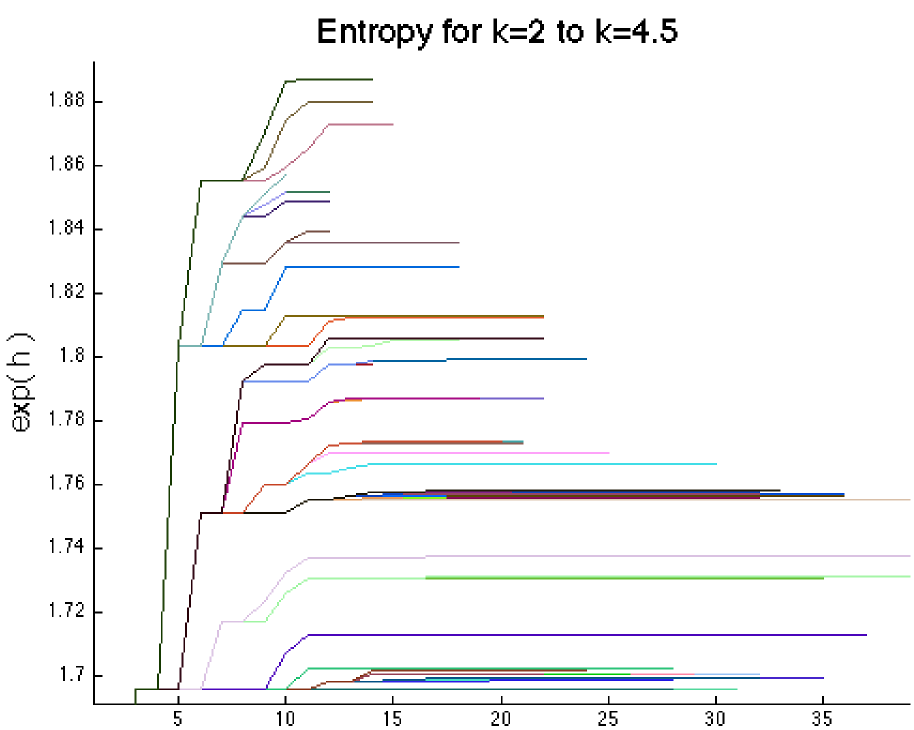


Figure 17 Random color entropy plot

entropy. Because the Hénon map is chaotic, this behavior was not all that unexpected, but it useful to tell where changes occur.

Graphing topological entropy in this way allowed us to see the entropy front pattern. Although we could tell that the entropy increased quicker for higher *k* values, we can now look at specific areas where the entropy is changing more rapidly. One grouping in particular, from k = 3.35 to 3.73, seems to stem from the period two orbit becoming large enough to influence the system as a whole early on. The second group, k = 2.55 to 3.17, may also arise from a periodic orbit, although we don't know if this is the case.

3.2 Errors

There were three main types of errors that occurred while running simulations. The first was an error that caused the simulation to stop running prematurely. This error did not affect any of the results significantly. The second error was caused when the simulation resulted in intersections between two pieces of unstable manifold. Although

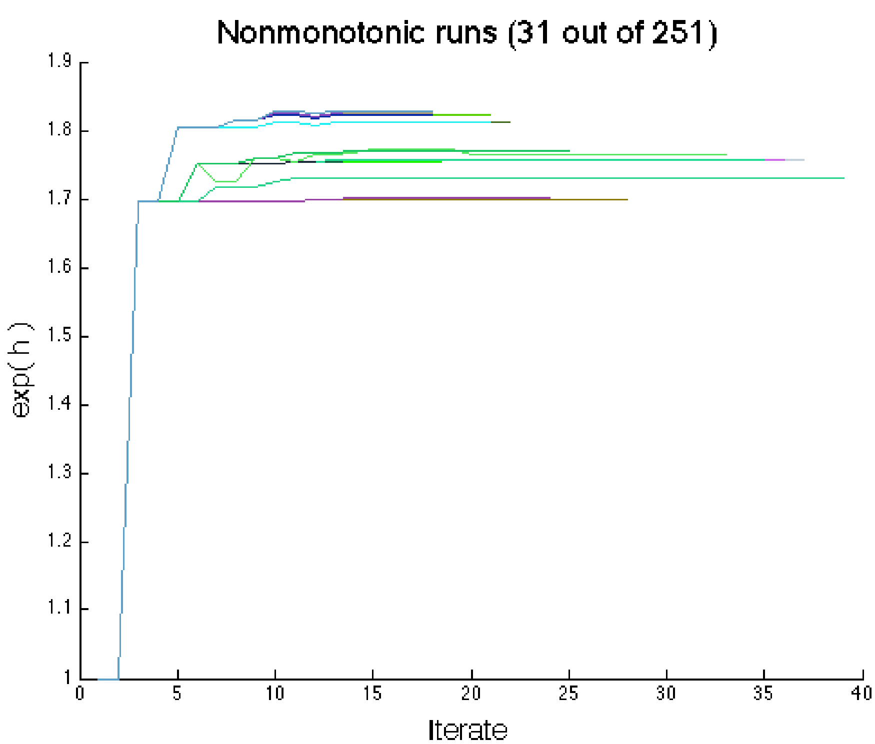


Figure 18 Nonmonotonic runs

this did not appear to affect the topological entropy, it did affect other quantities such as the number of symbols. The third type of error that occurred was that the entropy values had some nonmonotonic sections. This usually occurred for one or two consecutive values per k value that contained errors. Some of these nonmonotonic runs appear to follow the same profile as a nearby monotonic run, and in these cases it appears that only one value was incorrectly calculated. Other nonmonotonic runs were very small (less than 10^{-13}) and were probably the result of round-off error during simulation. Overall, there were 31 nonmonotonic runs out of 251 total runs that cannot be attributed to round-off error.

3.3 Future directions

There are many more potential areas to study in the Hénon map. One step that would improve this research would be finding out what causes the nonmontonicity and other errors. In addition, improving the matlab code to run more efficiently would allow for quicker analysis so that data could be taken more easily. However, this is a relatively minor issue, and more important aspects of this project can still be focused on in the future.

First, entropy fronts can be explored in more detail. Although there is a general trend of starting at late iterates for smaller k and coming in at early iterates for larger *k* values, more *k* values could be explored to see if this trend holds or if new patterns emerge. Also, the groups where entropy fronts come in rapidly, such as k = 2.55 to 3.17 group and the 3.35 to 3.73 group, could be looked at more closely to see if they correlate with the appearance of new periodic orbits. A larger *k* range could be inspected if some correlations are found to see if they hold for more values.

Secondly, the same *k* range could be reevaluated with a smaller area threshold to further explore late iterates. Allowing late-time behavior could show additional structure, especially for those values that showed no addition structure for all iterates examined. It could also reveal more information on higher *k* values since increasing the area threshold sometimes reduces the errors in computation that cause the code to stop early. If new complexities are then discovered, these may also be correlated to other values.

Third, the topological entropy could be computed while taking into account both inner and outer tangles (figure 19). Using one or more inner tangle could result in the lower bound entropy value increasing and converging faster. A higher area threshold or more robust code would be needed., but studying the Hénon map this way could show how period two or three tangles affect the dynamics of the Hénon map in more detail.

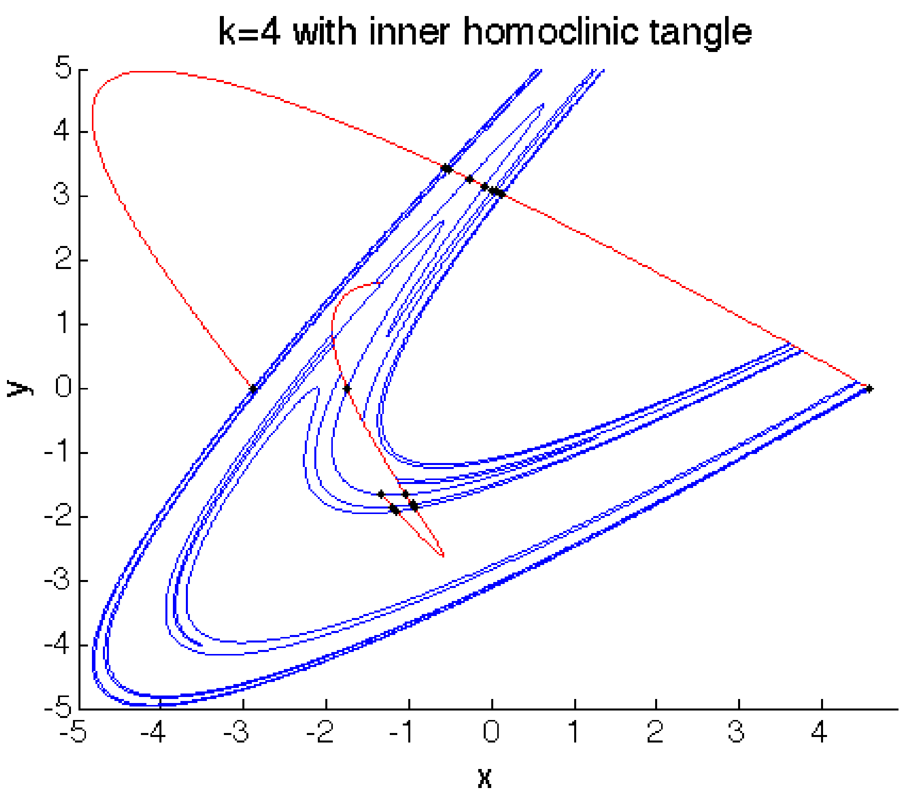


Figure 19 Hénon map with inner and outer tangles.

Finally, other information from the Hénon map could be compared over a similar range of *k* values. Because new periodic orbits occur quite often, trends in the appearances of these periodic orbits could be observed. Another measure of complexity might be the minimum number of symbols needed to describe the map at a given iterate. The current code used obtains much more information than the topological entropy alone, and this information could hold a key to discovering patterns in the behavior of the Hénon map. If discovered, these patterns could eventually lead to generalities that apply to more realistic models and could possibly lead to a deeper understanding of chaotic systems. Additionally, the Hénon map and other simple maps could be used as a test model to verify new theories about dynamical systems that may arise. These advancements would likely be some time in the future, but even a simple model like the Hénon map still may help find them.

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