ONLY USE THESE SOLUTIONS AFTER YOU HAVE TRIED TO ANSWER THE PROBLEMS ON YOUR OWN! Otherwise, you will not benefit from these solutions.

1. Find the surface area of the surface $z = \frac{2}{3}\,(x^{3/2} + y^{3/2})$ over the region $0 \le x \le 2$ and $0 \le y \le 1$.

   Solution
   Parametrize the surface with $x = x, y = y, z = \frac{2}{3}\,(x^{3/2} + y^{3/2})$.
   The surface area element is $dS = \sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + x + y}\; dx\; dy$
   The surface area is

$$
\begin{aligned}
A &= \int_0^2 dx \int_0^1 dy \sqrt{1+x+y} \\
&= \int_0^2 dx \frac{2}{3}(1+x+y)^{3/2}\,\Big|_{y=0}^{1} \\
&= \int_0^2 dx \frac{2}{3}\left((2+x)^{3/2} - (1+x)^{3/2}\right) \\
&= \frac{2}{3}\frac{2}{5}\left((2+x)^{5/2} - (1+x)^{5/2}\right)\Big|_{x=0}^{2} \\
&= \frac{4}{15}\left(4^{5/2} - 3^{5/2} - 2^{5/2} + 1^{5/2}\right) = \frac{4}{15}\left(33 - (3^{5/2} + 2^{5/2})\right)
\end{aligned}
$$

2. Compute the mass of a pringle shaped like the surface $z = x^2 - y^2$ over the region $x^2 + y^2 \le 1$ if its density (mass per unit area) is given by $d(x, y, z) = \sqrt{1 + 4z + 8y^2}$.

   Solution
   Parametrize the surface with $x = x, y = y, z = x^2 - y^2$.
   The surface area element is $dS = \sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + 4x^2 + 4y^2}$
   The mass is

$$
\begin{aligned}
M &= \int\int_{x^2+y^2\le 1} \sqrt{1+4(x^2-y^2)+8y^2}\sqrt{1+4x^2+4y^2}\; dx\; dy \\
&= \int\int_{x^2+y^2\le 1} \sqrt{1+4x^2+4y^2}\sqrt{1+4x^2+4y^2}\; dx\; dy \\
&= \int\int_{x^2+y^2\le 1} (1+4x^2+4y^2)\; dx\; dy \\
&= \int_0^{2\pi}\int_0^1 (1+4r^2) r\; dr\; d\theta \\
&= 2\pi(r^2/2 + r^4)\,\Big|_{r=0}^{1} \\
&= 2\pi(1/2 + 1) = 3\pi
\end{aligned}
$$

3. Compute the surface integral $\int\int_S \vec{F} \cdot \hat{n} dS$ over the surface S consisting of the portion of a cylinder of radius 3 centered on the y-axis between $y = 0$ and $y = 2$, $\hat{n}$ is pointing away from the cylinder and $\vec{F} =< xy, e^z, yz >$.

Solution
First approach: Direct integration
We parametrize the cylinder using: $x = 3\cos\theta, y = y, z = 3\sin\theta$ for $0 \leq \theta < 2\pi$ and $0 \leq y \leq 2$.
On the surface of the cylinder, the outer normal is $\hat{n} =< \cos\theta, 0, \sin\theta >$
The area element is $dS = 3\ d\theta\ dy$.
Our integrand is then $\vec{F} \cdot \hat{n}\ dS = (xy\cos\theta + yz\sin\theta)3\ d\theta\ dy$.
So out integral becomes

$$\begin{aligned}
\int\int_S \vec{F} \cdot \hat{n} dS &= \int_0^2 \int_0^{2\pi} (3y(\cos\theta)^2 + 3y(\sin\theta)^2)3\ d\theta\ dy \\
&= = \int_0^2 \int_0^{2\pi} 9y\ d\theta\ dy \\
&= (2\pi)(9y^2/2)|_0^2 = (2\pi)(9/2)(4) = 36\pi
\end{aligned}$$

Second approach: divergence theorem If we want to use the divergence theorem, we will need to close the surface. To do so, let's use flat disks at both ends.
On the disk D_1 of radius 3 in the xz-plane where $y = 0$, we have $\hat{n} =< 0, -1, 0 >$ and so $\vec{F} \cdot \hat{n} = -e^z$.
On the disk D_2 of radius 3 in the xz-plane where $y = 2$, we have $\hat{n} =< 0, 1, 0 >$ and so $\vec{F} \cdot \hat{n} = e^z$.
The divergence is $\nabla \cdot \vec{F} = y + 0 + y = 2y$.
Overall, we have

$$\begin{aligned}
\int\int_S \vec{F} \cdot \hat{n}\ dS &= \int\int_{S+D_1+D_2} \vec{F} \cdot \hat{n}\ dS - \int\int_{D_1} \vec{F} \cdot \hat{n}\ dS - \int\int_{D_2} \vec{F} \cdot \hat{n}\ dS \\
&= \int\int\int_V \nabla \cdot \vec{F}\ dV - \int\int_{D_1} \vec{F} \cdot \hat{n}\ dS - \int\int_{D_2} \vec{F} \cdot \hat{n}\ dS
\end{aligned}$$

where we have used the divergence theorem on the closed surface made up of S, D_1, D_2. Because the integrands in the last 2 integrals are equal and opposite and the domain, once parametrized, is the same, the last 2 integrals will cancel each other out. So we have

$$\begin{aligned}
\int\int_S \vec{F} \cdot \hat{n} dS &= \int\int\int_V 2y\ dV \\
&= \int_0^{2\pi} \int_0^3 \int_0^2 2\ y\ dy\ r\ dr\ d\theta \\
&= 2\pi \quad r^2/2|_0^3 \quad y^2|_0^2 = 2\pi(9/2)(4) = 36\pi
\end{aligned}$$