Lecture 9, Section 3.3: Arclength

Great, so we can do calculus component-by-component, but what does this mean?

For the limit, one can way to think of this is to think of t as time. Then the limit as $t \to t_0$ of $\vec{r}(t)$ is asking: Where were you as the time approached t_0? (You have the right to remain silent though)

The integral is mostly meaningful as an antiderivative really.

The derivative is the important one. Say $\vec{r}(t)$ represents a position changing in time. Then
$\frac{d\vec{r}}{dt}$ = rate of change of the position over time
which is the same as saying how fast is the position changing
which is the VELOCITY vector.
If you trace the curve $\vec{r}(t)$, then the velocity is TANGENT to the curve:

$$\frac{d\vec{r}(t)}{dt} = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

tangent vector

We call $|\vec{r}'(t)|$ the speed (it is a scalar). We also introduce

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

the unit tangent vector, which has length one and points in the direction of motion.
Note that we have $\int \vec{r}\,'(t)dt = \vec{r}(t) + \vec{C}$

Now, what is $\vec{r}\,''(t)$? It is the ACCELERATION. It describes how $\vec{r}\,'(t)$ changes over time, both in magnitude and in DIRECTION. It is also a vector.

Example: $\vec{r}(t) =< \cos t, t, \sin t >$
This is a helix moving along the y-axis. We have
$\vec{v}(t) = \vec{r}\,'(t) =< -\sin t, 1, \cos t >$ with speed $|\vec{r}\,'(t)| = \sqrt{2}$ and
$\vec{a}(t) = \vec{r}\,''(t) =< -\cos t, 0, -\sin t >$.

Position, velocity, and acceleration

Arclength

How can we calculate the arclength of a curve?
Think of the curve as a string. There is an old folk tale where a God asked a warrior to measure the length of a sacred string, without touching it. The warrior's solution was to count how long it took for an ant to walk along it, and then multiply that by how fast the ant was walking! This is all true, and it works, except that there is no such old folk tale, I made it up.

So how fast does our "ant" go? $|\vec{r}'(t)|$ = speed. How far does it go if it walk for a time of Δt?
$\Delta s = |\vec{r}'(t)|\Delta t$.

Adding up small displacements to compute the total arclength

But usually the speed is not constant, so we need to add up all those small contributions:
Arclength is $s = \lim_{n\to\infty} \sum_{i=0}^{n} |\vec{r}'(t_i)|\Delta t_i = \int_0^t |\vec{r}'(\tau)| d\tau$
Or in more details:

$$s(t) = \int_0^t \sqrt{(x'(\tau))^2 + (y'(\tau))^2 + z'(\tau))^2} d\tau$$

What do you think is ds/dt? Well it is your speed again $|\vec{r}\,'(t)|$.

Often, we like to use s as a parameter instead of t, because it is a "natural" choice, that has more meaning than an arbitrarily chosen time. So we then have $\vec{r}(s)$ is a curve and for example $\vec{r}(1)$ is your position after you traveled one unit in distance.

What would be $\frac{d\vec{r}}{ds}$ then? and $|\frac{d\vec{r}}{ds}|$? That last one is 1, because your speed is then exactly 1, always. To calculate the derivative, we use the chain rule

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \vec{T}(t)$$

so it is the unit tangent vector again.

Example: Give a formula for the arclength of $\vec{r}(t) =< 5t, 7t^2, 4t^3 >$ between the points $(0, 0, 0)$ and $(10, 28, 32)$.

At the starting point, we see that $t = 0$. At the end point, using $x(t) = 5t = 10$, we see that $t = 2$. So the arclength formula will be applied, integrating from 0 to 2:

$$s = \int_0^2 \sqrt{5^2 + (14\tau)^2 + (12\tau^2)^2}\, d\tau = \int_0^2 \sqrt{25 + 196\tau^2 + 144\tau^4}\, d\tau.$$

This integral does not have a closed-form expression, but it can be approximated numerically (take Math 130/131 to know how!).

Note that a big deal if you study curves a bit more (like I do) is $\frac{d\vec{T}}{ds}$ and $\kappa = \left|\frac{d\vec{T}}{ds}\right|$. This last one is called the curvature, and we have $\frac{d\vec{T}}{ds} = \kappa\vec{n}$, where $\vec{n}$ is a unit vector perpendicular to the curve (and so is perpendicular to the unit tangent vector).