Quiz 2 taken today or Tuesday
Quiz 3 posted later today CHECK DATES on QUIZZES

Homework 8 posted, Worksheet 2 posted
Lecture 8 notes posted

Off. Hrs: Lucas today 1-3pm, ACS 362B

Lecture 8, Section 3.1, 3.2: Vector valued functions and space curves

We study functions that take one real number, say t,
and give a vector (like position $\vec{r}$, velocity $\vec{v}$, force $\vec{F}$, etc).

Such functions are called vector-valued functions: $\mathbb{R} \to \mathbb{R}^3$.
$t :\to (x(t), y(t), z(t))$
This can be thought of as a space curve, being traced over time t.

We already know some functions like this! Lines: $\vec{r}(t) =< 2, 3, 0 > +t < 1, -2, 1 >$ or
$x(t) = 2 + t$
$y(t) = 3 - 2t$
$z(t) = t$

These are parametric equations, like the ones you saw in 2D.
Remember this: $x(t) = 2t$, $y(t) = t^2$? This implies that $z = 0$ (and that $y = (x/2)^2$).

a 2D parametrized curve.

Another example: $x(t) = \cos t$, $y(t) = 3\sin t$, is the same as $x^2 + (y/3)^2 = 1$, an ellipse.

A second example of a parametrized curve.

Note: the same curve may be represented by several parametrisations, through any legitimate change of variables: For example $u = 3t - 1$.

We use the same idea in 3D, although it is sometimes harder to see.
Example: $x(t) = \cos t$, $y(t) = t$, $z(t) = \sin t$ is a helix along the y axis.

A parametrized 3D curve: a helix

Example: $x(t) = t$, $y(t) = t^2$, $z(t) = t^3$ is a cubish thingy.

A second parametrized 3D curve

One common occurrence is at the intersection of 2 surfaces: Intersection of $y^2 + z^2 = 16$ and $x + y = 1$.

A plane cutting through a cylinder

Try $y(t) = 4\cos t$, then $z(t) = 4\sin t$ from the first surface, and find $x(t) = 1 - 4\cos t$ from the second.

A plane cutting through an elliptical cone.

Intersection of $z^2 = 2x^2 + y^2$ and $z = x - 2$.
First, we eliminate a variable, in this case z:

$$\begin{aligned}
(x-2)^2 &= 2x^2 + y^2 \\
x^2 - 4x + 4 &= 2x^2 + y^2 \\
4 &= x^2 + 4x + y^2 \quad \text{now we complete the square} \\
4 &= (x+2)^2 - 4 + y^2 \\
8 &= (x+2)^2 + y^2
\end{aligned}$$

So in the variables x and y alone, we have a circle, which we parametrize as:
$x + 2 = \sqrt{8}\cos t$ and $y = \sqrt{8}\sin t$
and we can find z from $z = x - 2$, so that in the end we have:
$x = \sqrt{8}\cos t - 2$
$y = \sqrt{8}\sin t$
$z = \sqrt{8}\cos t - 4$.

We can now (finally!) do calculus on space curves. Really, it is easy: you do it component-by-component.

Say $\vec{r}(t) =< x(t), y(t), z(t) >$. Then we have

$$\begin{aligned}
\lim_{t\to 2} \vec{r}(t) &= < \lim_{t\to 2} x(t), \lim_{t\to 2} y(t), \lim_{t\to 2} z(t) > \\
\frac{d\vec{r}(t)}{dt} = \vec{r}'(t) &= < x'(t), y'(t), z'(t) > \\
\int \vec{r}(t)dt &= < \int x(t)dt, \int y(t)dt, \int z(t)dt >
\end{aligned}$$

There are differentiation rule, some old ones:

$$\begin{aligned}
(\vec{u}(t) \pm \vec{v}(t))' &= \vec{u}'(t) \pm \vec{v}'(t) \\
(c\vec{u}(t))' &= c\vec{u}'(t) \\
(\vec{u}(f(t)))' &= f'(t)\vec{u}'(f(t))
\end{aligned}$$

And some that are new, but they don't look very different from what you know

$$\begin{aligned}(f(t)\vec{u}(t))' &= f(t)\vec{u}'(t) + f'(t)\vec{u}(t)\\ (\vec{u}\times\vec{v})' &= \vec{u}'\times\vec{v} + \vec{u}\times\vec{v}' \text{ still a vector}\\ (\vec{u}\cdot\vec{v})' &= \vec{u}'\cdot\vec{v} + \vec{u}\cdot\vec{v}' \text{ still a scalar}\end{aligned}$$

We can easily check that last one, say in 2D:

$$(\vec{u}\cdot\vec{v})' = (u_xv_x + u_yv_y)' = u_x'v_x + u_xv_x' + u_y'v_y + u_yv_y' = u_x'v_x + u_y'v_y + u_xv_x' + u_yv_y' = \vec{u}'\cdot\vec{v} + \vec{u}\cdot\vec{v}'$$

That last one is useful when considering lengths. Consider a satellite in orbit around the earth. Say its distance to the center of the earth is constant: $|\vec{r}(t)| = R$. Then we have

$$\begin{aligned}\vec{r}(t)\cdot\vec{r}(t) &= R^2\\ \frac{d(\vec{r}(t)\cdot\vec{r}(t))}{dt} &= 0\\ \vec{r}'\cdot\vec{r} + \vec{r}\cdot\vec{r}' &= 0\\ 2\vec{r}'\cdot\vec{r} &= 0\\ \vec{r}'\cdot\vec{r} &= 0\end{aligned}$$

That means that the position (relative to the origin) is always perpendicular to the velocity $\vec{r}'$. This is true with higher derivatives too, so the velocity is perpendicular to the acceleration.

Position is perpendicular to velocity.