Lecture 5, Section 2.5: Lines and Planes

Lines:

Say we pick a vector $\vec{v}$. Let's make the length of $\vec{v}$ variable, by multiplying $\vec{v}$ by a parameter t, so we have $t\vec{v}, t \in \mathbb{R}$.

A point and a vector describe a line

Starting at a point P, we now add a multiple of $\vec{v}$: $\vec{r} = P + t\vec{v}$. If we allow t to take any value, what do I get? A line. Coordinate by coordinate, we get:

$$\begin{aligned} x(t) &= P_x + tv_x \\ y(t) &= P_y + tv_y \\ z(t) &= P_z + tv_z \end{aligned}$$

Example: Find the line going through $P = (-1, -2, -3)$ and $Q = (2, 0, 1)$.
The direction vector is $\vec{PQ} =< 3, 2, 4 >$. So we get

$$\begin{aligned} x_1(t) &= -1 + 3t \\ y_1(t) &= -2 + 2t \\ z_1(t) &= -3 + 4t \end{aligned}$$

If $t = 0$, we are at P. If $t = 1$, we are at Q, and by varying t we can cover the entire infinite line.

Does it intersect the following line?

$$\begin{aligned} x_2(s) &= 3 + s \\ y_2(s) &= 2s \\ z_2(s) &= -1 - s \end{aligned}$$

Or is it parallel to it?
The direction vector here is $< 1, 2, -1 >$, and that is not parallel (not a multiple) to $\vec{PQ}$. We look for an intersection:

$$x_1(t) = x_2(s) \text{ so } -1 + 3t = 3 + s \text{ so } s = -4 + 3t$$

Now we use the other coordinates to find s

$$y_1(t) = y_2(s) \text{ so } -2 + 2t = 2s = -8 + 6t \text{ so } t = 3/2, s = 1/2$$

In 2D we would be done, line are either parallel or they intersect. In 3D, we need to check the third coordinate

$$z_1(t) = -3 + 4t = 3 \text{ and } \quad z_2(s) = -1 - s = -3/2$$

The z coordinate is not the same, so they DON'T intersect.

Note that the lines may be given with the same parameter (both with t for example). YOU must change one of the parameters then.

One can also eliminate t to get the so-called symmetric equations

$$t = \frac{x - P_x}{v_x} = \frac{y - P_y}{v_y} = \frac{z - P_z}{v_z}$$

Planes:

We want an equation to describe all the points in a plane. To specify a unique plane, we need a point P and a vector NORMAL to the plane $\vec{n}$. How do we know if $\vec{R} = (x, y, z)$ is in the plane?

A plane and its normal vector.

Let $P = (P_x, P_y, P_z)$ and $\vec{n} = (n_x, n_y, n_z)$.
Then $\vec{PR} =< x - P_x, y - P_y, z - P_z >$
If R is in the plane, $\vec{PR} \perp \vec{n}$ so $\vec{PR} \cdot \vec{n} = 0$. So

$$n_x(x - P_x) + n_y(y - P_y) + n_z(z - P_z) = 0$$

or equivalently

$$xn_x + yn_y + zn_z - (n_xP_x) + n_yP_y + n_zP_z) = 0$$

which in general we write as

$$ax + by + cz + d$$

with x, y, z variables and $a, b, c,$ and d constants.

Example: What plane goes through $P = (2, 0, 1)$ with a normal given by $\vec{n} =< 3, 2, -1 >$?
Using the method above, we find that $3x + 2y - z - 5 = 0$ is the plane Π.

Quiz 1 taken today or Tuesday
Quiz 2 posted later today CHECK DATES on QUIZZES

Homework 5 posted, Worksheets posted
Lecture 5 notes posted

Off. Hrs: Lucas today 1-3pm, ACS 362B

Calculating the distance between a point and a plane.

What is the (minimum) distance between $P_1 = (3, 1, 6)$ and Π? Plugging this in, we see that it satisfies the given equation. So P_1 is actually ON the plane, and so its distance to the plane is 0.

What is the distance between $P_2 = (4, 1, 6)$ and Π?
Find the length of the projection of $\vec{PP_2}$ onto $\vec{n}$:
$\vec{PP_2} =< 2, 1, 5 >, \vec{n} =< 3, 2, -1 >$.
distance = $\cos\theta|\vec{PP_2}| = \frac{\vec{PP_2}\cdot\vec{n}}{|\vec{n}|} = \frac{3}{\sqrt{14}}$ (less than 1).

Distance from a line to a plane? Either 0 or pick any point on the line and to what we just did, since the line must then be parallel to the plane.
Distance from a plane to a plane? Either 0 or pick any point in one of the planes and compute its distance to the other plane.
Distance from a line to a line? Either 0 or you need to make two planes with normal $\vec{n} = \vec{r_1} \times \vec{r_2}$ and each containing one line. They are now parallel, and we can use the method from the previous line.

How do you find the angle between two planes:
$\Pi = 3x + 2y - z - 5 = 0$ and
$\chi = 2x - y + 2z - 3 = 0$ Well it is the same as the angle between their normals so:

The angle between two planes is the same as that between their normals.

$\vec{n_\Pi} =< 3, 2, -1 >$ and $\vec{n_\chi} =< 2, -1, 2 >$.
$\cos\theta = \frac{\vec{n_\Pi}\cdot\vec{n_\chi}}{|\vec{n_\Pi}||\vec{n_\chi}|} = \frac{2}{\sqrt{14}\sqrt{9}}$

What is the equation of the line intersecting them? We need a point and a vector. A point P on both planes must satisfy both equations: $3x + 2y - z - 5 = 0$ and
$2x - y + 2z - 3 = 0$
Try $z = 0$, this leaves us with $3x + 2y = 5$ and $2x - y = 3$. So $y = 2x - 3$, $7x = 11$ and $x = 11/7, y = 1/7$.
So $P = (11/7, 1/7, 0)$.
The direction vector has to be perpendicular to both normals. So it is $\vec{r} = \vec{n_\Pi} \times \vec{n_\chi}$

$$\vec{r} = \vec{n_\Pi} \times \vec{n_\chi} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -1 \\ 2 & -1 & c2 \end{vmatrix} = 3\vec{i} - 8\vec{j} - 7\vec{k} =< 3, -8, -7 >$$

So the line is $x(t) = 3t + 11/7, y(t) = -8t + 1/7, z(t) = -7t$.