Quiz 1 taken next Friday or Tuesday

LOOK AT DATES on QUIZZES

Homework 4 posted, Worksheets posted
Lecture 4 notes posted

Buy Green/Blue books

Off. Hrs: Friday 9am, ACS 362C

Lecture 4, Section 12.4: Cross product

Recall the dot product: scalar = $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = |\vec{a}||\vec{b}| \cos\theta$.

There is a second product between 2 vectors: the cross product. This one results in a vector: $\vec{v} \times \vec{w} = \vec{z}$.

Basic properties (see page 153):
$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ (anti-commutativity, UNUSUAL)
$(s\vec{a}) \times \vec{b} = \vec{a} \times (s\vec{b}) = s(\vec{a} \times \vec{b})$ (associativity)
$\vec{a} \times \vec{a} = 0$
$(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$ (distributivity)

Geometric properties:
1) $\vec{z} \perp \vec{v}$ and $\vec{z} \perp \vec{w}$ (orthogonality)
2) $\vec{z}$ points in the direction given by the right-hand rule: $\vec{v}$ = index, $\vec{w}$ = major, $\vec{z}$ = thumb
3) Length of $\vec{z}$ is the area of the parallelogram generated by $\vec{v}$ and $\vec{w}$.
4) $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\theta$.

Property 3 is equivalent to property 4

How do you compute $\vec{v} \times \vec{w}$?
Formula: $\vec{v} \times \vec{w} = (v_y w_z - v_z w_y)\hat{\imath} + (v_z w_x - v_x w_z)\hat{\jmath} + (v_x w_y - v_y w_x)\hat{k}$
To remember that, we introduce the determinant:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = (v_y w_z - v_z w_y)\hat{\imath} + (v_z w_x - v_x w_z)\hat{\jmath} + (v_x w_y - v_y w_x)\hat{k}$$

So why does this formula have properties 1, 2, 3 and 4?
We can check 1) directly: $\vec{v} \cdot (\vec{v} \times \vec{w}) = v_x(v_y w_z - v_z w_y) + v_y(v_z w_x - v_x w_z) + v_z(v_x w_y - v_y w_x) = 0$.
And the same goes for $\vec{w}$. For property 2), you can try one case, and then get all other cases via rotation (which doesn't change the cross product).

Properties 3) and 4) are the same really:

$$\begin{aligned}
|\vec{v}\times\vec{w}|^2 &= (v_y w_z - v_z w_y)^2 + (v_z w_x - v_x w_z)^2 + (v_x w_y - v_y w_x)^2 \\
&= (v_x^2+v_y^2+v_z^2)(w_x^2+w_y^2+w_z^2) - (v_x w_x + v_y w_y + v_z w_z)^2 \\
&= |\vec{v}|^2|\vec{w}|^2 - (\vec{v}\cdot\vec{w})^2 \\
&= |\vec{v}|^2|\vec{w}|^2 - |\vec{v}|^2|\vec{w}|^2\cos^2\theta \\
&= |\vec{v}|^2|\vec{w}|^2(1-\cos^2\theta) \\
&= |\vec{v}|^2|\vec{w}|^2\sin^2\theta
\end{aligned}$$

Applications:
1) Area of a triangle or parallelogram
$\vec{a} =< 2,1,0 >$ and $\vec{b} =< 1,3,0 >$ Area of the parallelogram they span: $| < 0,0,5 > | = 5$

Computing Area with the cross-product

Area of the triangle formed by joining their ends: 5/2.

2) Finding a vector perpendicular to 2 given vectors (very important one!)
Take 3 points in space. They must lie within a plane. How can we find a vector that is normal (perpendicular) to that plane? Points are $P =< 1,2,-1 >$, $Q =< 3,5,0 >$ and $R =< 0,2,1 >$

Cross-product gives a vector perpendicular to BOTH the vectors used to compute it.

Form vectors: $\vec{PQ} = Q - P =< 2,3,1 >$ and $\vec{PR} = R - P =< -1,0,2 >$
The normal is $\vec{n} = \vec{PQ}\times\vec{PR} =< 6,-5,3 >$
Note that $\vec{n}\perp\vec{PQ}$ and $\vec{n}\perp\vec{PR}$. More generally, $\vec{n}\perp\vec{PX}$ for any point X in the plane.

We can now define the scalar triple product (not as critical as the dot or cross product, but still useful):

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Consider the box spanned by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ The height is $|\vec{a}|\cos\theta$ (θ is the angle to the vertical)

Squished box, of volume given by the triple product

The area of the bottom is $|\vec{b}\times\vec{c}|$
So the volume is $|\vec{b}\times\vec{c}||\vec{a}|\cos\theta = |\vec{a}\cdot(\vec{b}\times\vec{c})|$.

Important point: When is the triple product equal to 0?
If $\vec{a}\perp$ to $\vec{b}\times\vec{c}$
So if $\vec{a}$ is in the same plane as $\vec{b}$ and $\vec{c}$, in which case the volume of the box is 0.

With the same $\vec{PQ} =< 2,3,1 >$ and $\vec{PR} =< -1,0,2 >$ as before, is the point $S = (1,1,3)$ in the same plane as P, Q, and R?
Try $\vec{PS} =< -1,-2,2 >$.
$\vec{PS}\cdot\vec{n} = -6 + 10 + 6 = 10$ so it is out of the plane.