Student evaluations close Friday (14% response now)

Survey on immigration related stressors, posted

LAs -> talk to Keith Thompson

Final exam: You will not be allowed in after 8:30am

Lecture 38, Section 6.8: Divergence Theorem

Recall the definition of the Divergence:

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

I claimed at the time that this was measuring (Flux out - Flux in) at a point. Let's see why.

Practice quiz (12) posted -> NOT TO BE TAKEN
Solutions posted at the end of today.

Today: Divergence theorem (6.8) for flux integrals
Lecture 38
Homework 38
Office hours: today, Hannah, 1:30-3:30pm, ACS-312 - Friday, François, 9am-10am, ACS-362B

Flux coming out of a small cube centered at the origin.

Final exam in 7 days

Consider a box with sides of length Δx, Δy, and Δz. Let's compute the flux out of faces 1 and 3, with normals given by
$\vec{n}_1 =< -1, 0, 0 >$ and $\vec{n}_3 =< 1, 0, 0 >$, respectively.

On surface 1, we have the following parametrization: $x = -\Delta x/2, y = y, z = z$.
So $\vec{F} \cdot \vec{n}$ = -$F_1(\Delta x/2, y, z)$ and dA = dy dz, and the flux is

$$\text{Flux}_1 = \int_{-\Delta z/2}^{\Delta z/2} \int_{-\Delta y/2}^{\Delta y/2} -F_1(-\Delta x/2, y, z) \text{dy dz} \approx \ \Delta z \Delta y \ - F_1(-\Delta x/2, 0, 0)$$

Similarly, on surface 3, we have the following parametrization: $x = \Delta x/2, y = y, z = z$.
So $\vec{F} \cdot \vec{n}$ = $F_1(\Delta x/2, y, z)$ and dA = dy dz, and the flux is

$$\text{Flux}_3 = \int_{-\Delta z/2}^{\Delta z/2} \int_{-\Delta y/2}^{\Delta y/2} F_1(\Delta x/2, y, z) \text{dy dz} \approx \ \Delta z \Delta y \ F_1(\Delta x/2, 0, 0)$$

So that together, we have:

$$\begin{aligned} \text{Flux}_1 + \text{Flux}_3 &= \Delta z \Delta y \ (F_1(\Delta x/2, 0, 0) - F_1(-\Delta x/2, 0, 0)) = \Delta x \Delta z \Delta y \ \frac{F_1(\Delta x/2, 0, 0) - F_1(-\Delta x/2, 0, 0)}{\Delta x} \\ &\approx \Delta V \frac{\partial F_1}{\partial x}(0, 0, 0) \end{aligned}$$

On the other faces of the cube, we find something similar, so over the whole cube, the flux outward is:

$$Flux = \Delta V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) = \Delta V \text{ div} \cdot \vec{F}$$

So the divergence of $\vec{F}$ is a local FLUX DENSITY.

This leads us to the divergence theorem:
If $\vec{F}$ is differentiable and S is a smooth, CLOSED surface, then

$$\int\int_S \vec{F}\cdot\hat{n}dA = \int\int\int_V \mathrm{div}\vec{F}dV$$

where V is the INTERIOR of S and $\hat{n}$ points outward.

Illustration of the divergence theorem.

Example: $\vec{F}(x,y,z) = (x + x^2 + 1 + y)\vec{i} + (3y - 2xy + 4z)\vec{j} + (z^2 + e^{xy})\vec{k}$. What is the flux out of the sphere of radius 2 centered at the origin?

Flux coming out of a sphere of radius 2 centered at the origin.

We could parametrize S, plug in, integrate. It would be long. OR

$$\int\int_S \vec{F}\cdot\hat{n}dA = \int\int\int_V \mathrm{div}\vec{F}dV$$

div$\vec{F}$ = 1 + 2x+3-2x+ 2z = 4 + 2z. So the flux is:

$$\int\int\int_{Sphere}(4+2z)dV = 4\left(\frac{4\pi 2^3}{3}\right) + 0 \text{ by symmetry}$$

So the flux is $128\ \pi/3$

Last example: Let S be the surface of a half-sphere $x^2+y^2+z^2=9$, $y \leq 0$ and consider a normal pointing away from the sphere. If $\vec{F} = (4x - z^2, (y-1)^2, \sin y + z)$, what is $\int\int_S \vec{F}\cdot\hat{n}dA$?

Using the divergence theorem for a half-sphere

We note that div $\vec{F} = 2 + 2(y-1) + 1 = 3 + 2y$, which is nice and simple.

S is not closed, so we can't use the divergence theorem, at least not directly.
We can close the half-sphere though. Let D be the disk $x^2+z^2 \leq 9$, $y = 0$. Then $S+D$ is a closed half-sphere. The divergence theorem then says

$$\begin{aligned}
\int\int_{S+D} \vec{F}\cdot\hat{n}dA &= \int\int\int_V \text{div } \vec{F} dV \\
&= \int_0^{\pi}\int_{\pi}^{2\pi}\int_0^3 (3 + 2\rho\sin\theta\sin\phi)\rho^2\sin\phi \, d\rho\, d\theta\, d\phi \\
&= 3V + \int_0^{\pi}\int_{\pi}^{2\pi}\int_0^3 (2\rho^3\sin\theta\sin^2\phi)\, d\rho\, d\theta\, d\phi \\
&= \frac{3}{2}\frac{4\pi 3^3}{3} + 2\frac{3^4}{4}(-\cos\theta)\big|_{\pi}^{2\pi}\left(\frac{\pi}{2}\right) \\
&= 54\pi - 81\pi/2 = (27/2)\pi
\end{aligned}$$

We we are really after is

$$\int\int_S \vec{F}\cdot\hat{n}dA = \int\int_{S+D} \vec{F}\cdot\hat{n}dA - \int\int_D \vec{F}\cdot\hat{n}dA$$

On D we have $\hat{n} =< 0, 1, 0 >$, and $y = 0$, so we find

$$\int\int_S \vec{F}\cdot\hat{n}dA = (27/2)\pi - \int\int_D (y-1)^2 dA = (27/2)\pi - \int\int_D (-1)^2 dA = (27/2)\pi - 9\pi = (9/2)\pi$$

And that is our FINAL answer.