Survey on immigration stresses

MATH 23: Multi-variable Calculus *Spring Semester 2026*

LAs -> talk to Keith Thompson

Lecture 37, Section 6.6c: Surface Integrals: Flux

Student evaluations -> please take a moment

We now consider flux integrals, which measure how much of a certain vector field, F, is flowing through a certain surface S.

Recall that we had:
$\vec{r}(u,v) =< x(u,v), y(u,v), z(u,v) >$
$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{||\vec{r}_u \times \vec{r}_v||}$ and
$dS = ||\vec{r}_u \times \vec{r}_v||dudv$.

A special type of integrand is the FLUX through a surface. If we let $\vec{F}(x,y,z) =< F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) >$ be a velocity field, and $\hat{n}$ be a unit normal to the surface S.

Last quiz (11) taken Friday or Tomorrow
Practice quiz (12) posted -> NOT TO BE TAKEN

Today: Flux integrals
Lecture 37
Homework 37
Office hours: François, 1-2pm, ACS-362B

Final exam in 9 days

The flux is the amount of $\vec{F}$ that crosses a surface.

The product $\vec{F} \cdot \hat{n}$ is the component of $\vec{F}$ that is parallel to $\hat{n}$, which is to say the part of $\vec{F}$ that goes THROUGH the surface S. In general we have at a point on the surface that

$$\vec{F} = (\vec{F} \cdot \hat{n})\hat{n} + \text{ a vector tangent to } S$$

Example: $\vec{F} =< -x+y, -x-y, -z >$ is the amount of solar wind in space, per unit area, with $(0,0,0)$ being at the center of the Earth. How much solar wind enters the atmosphere? Let's say that the atmosphere has radius 7 (thousand km).

Example: solar wind into the atmosphere.

Solar wind in = $\int\int_S \vec{F} \cdot \hat{n} dS$, with $\hat{n}$ a normal pointing inward. This becomes

$$\int_0^{\pi} d\phi \int_0^{2\pi} d\theta < -7\cos\theta\sin\phi + 7\sin\phi\sin\theta, -7\cos\theta\sin\phi - 7\sin\theta\sin\phi, -7cos\phi > \cdot < -\cos\theta\sin\phi, -\sin\theta\sin\phi, -\cos\phi > 49\sin$$

because $\vec{r}(\theta,\phi) =< 7\cos\theta\sin\phi, 7\sin\theta\sin\phi, 7\cos phi >$ and
$\vec{r}_\theta \times \vec{r}_\phi = 49\sin\phi < \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi > (-1)$

This integral simplifies to

$$\int_0^{\pi} d\phi \int_0^{2\pi} d\theta 7^3 \sin(\phi)(\sin^2\phi + \cos^2\phi) = 7^3 2\pi 2 = 4\pi 7^3$$

You can also think of the flux as the amount of water going through a net, with $\vec{F}$ is the velocity field, and $\hat{n}$ the unit normal to the net. The product $\vec{F} \cdot \hat{n}$ is how much crosses the net.

There are a few common surfaces you are likely to encounter:
Sphere:
$\vec{r}(\theta, \phi) =< R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos phi >$
$\hat{n} =< \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi >$
$dS = R^2 \sin\phi d\phi d\theta$.

Cylinder (vertical)
$\vec{r}(\theta, z) =< R\cos\theta, R\sin\theta, z >$
$\hat{n} =< \cos\theta, \sin\theta, 0$
$dS = Rd\theta dz$.

Surface $z = f(x, y)$.
$\vec{r}(x, y) =< x, y, f(x, y) >$
$\hat{n} =< f_x, f_y, -1 >$
$dS = \sqrt{1 + f_x^2 + f_y^2} dxdy$.