Lecture 36, Section 6.6c: Surface Integrals

We have been integrating over a single surface so far, the xy-plane.

Last quiz (11) taken today or Tuesday
Practice quiz (12) posted later today -> NOT TO BE TAKEN

Today: Surface integrals
Lecture 36 (and finishing 35)
Homework 36
Office hours: Lucas, 1-3pm, ACS-362B

Final exam in 12 days

Second exam:
Mean: 55.66
Median: 52.75
Slightly up from exam 1

Same deal as exam 1:
Better grade on the final can
replace ONE unit exam
(I will pick the most helpful)

Student evaluations are up
Please take a moment and
be constructive for next time.

Integrating over a domain in the xy-plane.

Now we want our domain to be curved.

Integrating using a general surface as a domain.

For example, we might want to integrate a quantity (density of pollutants) over a sphere (the Earth).

To do this, we need to:
1) parametrize the surface with $\vec{r}(u, v)$
2) Write your integrand with u, v: $f(u, v) = f(x(u, v), y(u, v), z(u, v))$
3) write dA in terms of u and v.

Integrating over a domain given by a half-sphere.

We then have

$$\lim_{n\to\infty,m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m}\Delta A f(x_{i,j},y_{i,j},z_{i,j}) = \int\int_S f(u,v)dA = \int\int_S f(u,v)||\vec{r}_u\times\vec{r}_v||dudv$$

For example, on Earth, we would use spherical coordinates, with $\rho = R$ fixed. Say we wanted to integrated $f(x,y,z) = z^2$. The area element is $R^2\sin\phi d\phi d\theta$ and we have

$$< x,y,z >=< R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi >$$

So we get the integral

$$\int_0^{\pi} d\phi \int_0^{2\pi} d\theta R^2\cos^2\phi(R^2\sin\phi)$$

which give $4\pi R^4/3$.

Why was the surface area element $R^2\sin\phi$? We can get this from the volume element in spherical coordinates, or by taking $||\vec{r}_\theta\times\vec{r}_\phi||$.

If we consider the simplest parametrization, $x = u$, $y = v$, $z = f(u,v)$, which represents the surface $z = f(x,y)$, we find:
$\vec{r}_u =< 1,0,f_x >$ and
$\vec{r}_v =< 0,1,f_y >$
So $dA = ||\vec{r}_u\times\vec{r}_v||dudv = \sqrt{1+f_x^2+f_y^2}dudv$.
To integrate the function $g(x,y,z)$ over the surface $z = f(x,y)$, we then have

$$\int_{u=u_0}^{u=u_1}\int_{v=v_0}^{v=v_1} g(u,v,f(u,v))\sqrt{1+f_x^2+f_y^2}dudv$$

Later, we will integrate $\vec{F}\cdot\vec{n}$, to compute the flux through a surface.