Lecture 34-35, Section 6.6: Surface parametrization continued

Let's say you are given a parametrization $\vec{r}(u, v)$. What can you say about the vectors $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$?

If you fix $v = v_0$, as you do when taking a derivative with respect to u, you get a trace, a curve in space which is part of the surface $\vec{r}(u, v)$. The vector $\frac{\partial \vec{r}}{\partial u}$ will be tangent to that curve, and so tangent to the surface. Similarly, $\frac{\partial \vec{r}}{\partial v}$ is also tangent to the surface.

Tangent vectors to a parametrized surface.

So if $< x_u, y_u, z_u >$ and $< x_v, y_v, z_v >$ are tangent to the surface, how do you find a vector normal to the surface? By taking the cross product
$\vec{n} = \vec{r}_u \times \vec{r}_v$.
and a unit normal would be $\hat{n} = \vec{n}/||\vec{n}|| = \frac{\vec{r}_u \times \vec{r}_v}{||\vec{r}_u \times \vec{r}_v||}$.

Example: Given $\vec{r}(u, v) =< \cos u \sin v, 3 \sin u \sin v, 3 \cos v >$, with $0 \le u \le 2\pi$ and $0 \le v \le \pi$, an ellipsoid, what is its normal?
Well $\vec{r}_u =< -\sin u \sin v, 3 \cos u \sin v, 0 >$
and $\vec{r}_v =< \cos u \cos v, 3 \sin u \cos v, -3 \sin v >$
and their cross-product is $\vec{n} =< -9 \cos u \sin^2 v, -3 \sin u \sin^2 v, -3 \sin v \cos v >$ So given any u, v, $\vec{r}(u, v)$ gives a point on the surface and $\vec{n}$ gives a normal vector at that point.

Question: What would be the element of surface area along this surface?

Area element of a parametrized surface.

The area of the tangent plane at a point would be

$$\Delta A = \Delta u \Delta v ||\vec{r}_u \times \vec{r}_v||$$

This is the same approach we used to find the area element of a general change of coordinates, except that then $z = 0$ and here z can also depend on u and v.

So to calculate a surface area, we do:

$$A = \int\int_S dA = \int_{u=u_0}^{u=u_1} \int_{v=v_0}^{v=v_1} ||\vec{r}_u \times \vec{r}_v|| dv du$$

Example with a sphere, same as the ellipsoid from before, with coefficients of 3 for all coordinates. Then $\vec{r}_u \times \vec{r}_v =< -9\cos u \sin^2 v, -9\sin u \sin^2 v, -9\sin v \cos v >$.
The length of that is $9\sin v$, so the surface area of a sphere is

$$A = \int_0^{\pi} \int_0^{2\pi} 9\sin v du dv = 9 \cdot 2\pi(-\cos v|_0^{\pi}) = 9 \cdot 4\pi = 4\pi R^2$$

One last example of a parametrisation:

$$\begin{aligned} x(u,v) &= (3+\cos v)\cos u \\ y(u,v) &= (3+\cos v)\sin u \\ z(u,v) &= \sin v \end{aligned}$$

Parametrizing a donut/bagel/ring.

Note that if $u = 0$, we have $(x-3)^2 + z^2 = 1$ and $y = 0$. If $u = \pi/2$ we have $(y-3)^2 + z^2 = 1$ with $x = 0$.
If we fix v, we get circles or varying radii.

The whole thing is a doughnut.