Lecture 32, Section 6.5: Curl and divergence

We have met the curl before, in 2D, and briefly in 3D. It is

$$\text{curl}\vec{F} =< \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} >$$

and we saw that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ was a circulation density when $\vec{F}$ is a velocity field.

Here we give a more systematic way to calculate the curl and we interpret its meaning.

Introduce the differential operator ∇ or $\vec{\nabla} =< \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} >$.

We have met it before: grad $f = \nabla f =< \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} > f =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >$.

Now we can use it again:

$$\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

The physical meaning of the curl is that it measures how much $\vec{F}$ pushes a marker to rotate. Recall it is a circulation density.

Quiz 10 was Friday or will be Tuesday
No quiz after the exam, but there is a discussion.

Notes 32 today: Meaning of Curl and Divergence
Homework 32 posted

Office hours: today, François, 1-2pm, ACS 362B

Meaning of the curl: local rotation.

Exam 2 is Friday: please be prepared, on time, and ready to math for 50 minutes.

The direction of curl$\vec{F}$ is the rotation axis , using the right-hand rule, and $||\text{curl}\vec{F}||$ is the angular velocity.

Two special cases to keep in mind: Green's theorem

$$\oint \vec{F} \cdot d\vec{r} = \int \int_D \text{curl}\vec{F} \cdot \vec{k} dA$$

And if $\vec{F}$ is conservative, that is $\vec{F} = \nabla f$, then

$$\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

Well that is what we checked before (we called it 3 conditions, here it is one).

Second new operator: Divergence, div $\vec{F} = \nabla \cdot \vec{F}$
The definition is: $< \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} > \cdot < F_1, F_2, F_3 >= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$. It is a scalar, not a vector.

What does this mean? It measures how much of $\vec{F}$ is coming out of a point.
1) If div $\vec{F} > 0$, then more is coming out than going in.

Divergence as Flow out - Flow in.

2) If div $\vec{F} = 0$, then as much as coming in as in going out.
3) If div $\vec{F} < 0$, more is coming in than going out.

We will see why in §6.8.

So at a point (x_0, y_0, z_0) if $\vec{F}$ is a velocity field, then:
1) $\vec{F}(x_0, y_0, z_0)$ is how fast it moves.
2) curl $\vec{F}/||$curl $\vec{F}||$ is the axis about which it rotates.
3) $||$curl $\vec{F}||$ is how fast it rotates.
4) div $\vec{F}$ = going out - coming in.

One application: $\oint \vec{F} \cdot \vec{n} ds$ = how much is coming out of C. We have $\vec{F} =< F_1, F_2 >$.

Using Green's theorem to compute a flux.

$C = \vec{r}(t) =< x(t), y(t) >$
$\vec{n}(t) = \frac{<y'(t), -x'(t)>}{|d\vec{r}/dt|}$ so $\vec{n}ds =< y'(t), -x'(t) >$.
So we have $\vec{F} \cdot \vec{n}ds = (F_1 y' - F_2 x')dt =< -F_2, F_1 > \cdot < x', y' > dt$. So we get

$$\oint \vec{F} \cdot \vec{n}\, ds = \oint < -F_2, F_1 > \cdot \frac{d\vec{r}}{dt} dt = \int\int_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) dA = \int\int_D \text{div}\vec{F}\, dA$$

cool huh?