Lecture 30, Section 6.4: Green's theorem

Back to 2D: We study now non-conservative fields but with closed C.

Start simple Let $\vec{F} =< F_1, F_2 >$

A simple contour, to integrate in 2 ways.

We will approximate $\vec{F}$ on this square, taking $\Delta x, \Delta y \to 0$.
Then we have $F_1(x,y) \approx F_1(x_0,y_0) + (x-x_0)\frac{\partial F_1}{\partial x} + (y-y_0)\frac{\partial F_1}{\partial y}$.
Similarly $F_2(x,y) \approx F_2(x_0,y_0) + (x-x_0)\frac{\partial F_2}{\partial x} + (y-y_0)\frac{\partial F_2}{\partial y}$. Using this, we have a fairly simple integrals to take.
Along C_1, we have $\vec{r}(t) =< x_0 + (\Delta x)t, y_0 >$ for $0 \le t \le 1$.
So $\frac{d\vec{r}}{dt} =< \Delta x, 0 >$, and we get

$$\begin{aligned} \int_{C_1} \vec{F}\cdot\frac{d\vec{r}}{dt}dt &= \int_0^1 F_1(x_0+(\Delta x)t,\ y_0)\ \Delta x\ dt \approx && (1)\\ & \int_0^1 \left(F_1(x_0,y_0) + t\Delta x\ \frac{\partial F_1}{\partial x}\right)\Delta x dt = \Delta x F_1(x_0,y_0) + \frac{(\Delta x)^2}{2}\frac{\partial F_1}{\partial x} && (2)\end{aligned}$$

Along C_3, $\vec{r}(t) =< x_0 + \Delta x - (\Delta x)t, y_0 + \Delta y >$ with t going from 0 to 1. so we have

$$\begin{aligned} \int_{C_3} \vec{F}\cdot\frac{d\vec{r}}{dt}dt &= \int_0^1 F_1(x_0+\Delta x(1-t), y_0+\Delta y)dt \approx && (3)\\ & \int_0^1 \left(F_1(x_0,y_0) + \Delta x(1-t)\frac{\partial F_1}{\partial x} + \Delta y\frac{\partial F_1}{\partial y}\right)(-\Delta x)dt && (4)\\ &= (-\Delta x)F_1(x_0,y_0) + \frac{(-\Delta x)^2}{2}\frac{\partial F_1}{\partial x} + (-\Delta x\Delta y)\frac{\partial F_1}{\partial y} && (5)\end{aligned}$$

So $\int_{C_1+C_3} = -\Delta x\Delta y\frac{\partial F_1}{\partial y}$

Similarly, for the other 2 sides:
Along C_2, we have $\vec{r}(t) =< x_0 + \Delta x, y_0 + t\Delta y >$ for $0 \le t \le 1$.
So $\frac{d\vec{r}}{dt} =< 0, \Delta y >$, and we get

$$\begin{aligned} \int_{C_1} \vec{F}\cdot\frac{d\vec{r}}{dt}dt &= \int_0^1 F_2(x_0+\Delta x, y_0+t\Delta y)\ \Delta y dt \approx && (6)\\ & \int_0^1 \left(F_2(x_0,y_0) + (\Delta x)\frac{\partial F_2}{\partial x} + \Delta y\ t\frac{\partial F_2}{\partial y})\right)\Delta y dt && (7)\\ &= \Delta y F_2(x_0,y_0) + \frac{(\Delta y)^2}{2}\frac{\partial F_2}{\partial y} + \Delta x\Delta y\frac{\partial F_2}{\partial x} && (8)\end{aligned}$$

Along C_4, we get $-\Delta y F_2(x_0, y_0) - \frac{(\Delta y)^2}{2}\frac{\partial F_2}{\partial y}$ so we have
$\int_{C_2+C_4} = \Delta x \Delta y \frac{\partial F_2}{\partial x}$
So finally over the entire square, we get

$$\oint \vec{F} \cdot d\vec{r} = \Delta x \Delta y \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = dA \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

So the vorticity, $(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})$ is a circulation/area = circulation density!

What happens if I put small squares side-by-side?

Building a larger contour from a single square.

The common side cancels out, and we are left with only the outer boundary.

$$\oint_{S_1} \vec{F} \cdot d\vec{r} + \oint_{S_2} \vec{F} \cdot d\vec{r} + ... = \oint_C \vec{F} \cdot d\vec{r}$$

By taking the limit $\Delta x, \Delta y \to 0$, we can match any closed curve that way:

Building a general contour from squares.

and find

$$\oint_C \vec{F} \cdot d\vec{r} = \sum_{i=1}^{n}\sum_{j=1}^{m} \oint_{S_{i,j}} \vec{F} \cdot d\vec{r} = \sum_{i=1}^{n}\sum_{j=1}^{m} \Delta x \Delta y \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)|_{x_i, y_j} = \int\int_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

where D is the domain inside C. This is Green's theorem.

A few notes:
1) Here C is positively oriented (otherwise multiply by -1).

2)C is made of smooth pieces and CLOSED.
3) C doesn't cross itself.
4) Both $\frac{\partial F_2}{\partial x}$ and $\frac{\partial F_1}{\partial y}$ have to be continuous.

Important: Green's theorem works both ways: The two integrals are EQUAL, so we can compute whichever is the simplest.

Example: Compute $\oint \vec{F} \cdot d\vec{r}$, with C the unit circle $x^2 + y^2 = 1$ and $\vec{F} = -y\vec{i} + x\vec{j}$.

An example of usage of Green's theorem.

The curl is: curl$\vec{F} = \partial F_2/\partial x - \partial F_1/\partial y = 1 - (-1) = 2$. So we have

$$\oint \vec{F} \cdot d\vec{r} = \int\int_D 2dA = 2\pi(1)^2 = 2\pi$$

Example: find the area inside $x^2 + y^{2/3} = 1$. The area is $A = \int\int_D dA$. But here that is hard to do.

Using Green's theorem to compute surface area.

If we had a vector field $\vec{F}$ with a curl of 1, we could do

$$\int\int_D dA = \int\int_D \text{curl}\vec{F} dA = \oint \vec{F} \cdot d\vec{r}$$

Several vector fields are like that. Some simple ones are $\vec{F} = x\vec{j}$ and $\vec{F} = -y\vec{i}$.
Here parametrizing C is not so hard: $x(t) = \cos t, y(t) = \sin^3 t$ with $0 \leq t \leq 2\pi$. So we have

$$\frac{d\vec{r}}{dt} = < -\sin t, 3\sin^2 t \cos t >$$

And we can have $\vec{F} \cdot d\vec{r} = 3\sin^2 t \cos^2 t$ or $\sin^4 t$. Neither is so bad to integrate:

$$\int_0^{2\pi} 3\sin^2 t \cos^2 t dt = \int_0^{2\pi} \frac{3}{4}(\sin 2t)^2 dt = \frac{3\pi}{4} = \frac{3}{4}\left(\frac{t - \sin(4t)/2}{2}\right)\Bigg|_0^{2\pi}$$

One more example. Say $\vec{F} =< -yx^4 - x^2y^3/3, y^4x + x^3y^2/3 >$. Then the curl is $(y^2+x^2)^2$.
What is $\oint \vec{F} \cdot d\vec{r}$ where C_1 is the upper half circle of radius 1.

How to deal with an open contour? Close it!

We could parametrize , but the resulting integral is messy $\int_0^\pi (4/3)(\sin^2\theta\cos^4\theta + \sin^4\theta\cos^2\theta)d\theta$.
On the other hand, we could also close the contour with a line on the x-axis (figure). We would then have

$$\int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \int\int_D \text{curl}\vec{F} dA$$

So

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int\int_D \text{curl}\vec{F} dA - \int_{C_2} \vec{F} \cdot d\vec{r}$$

But on C_2, $\vec{F} =< 0, 0 >$. so all we need is to integrate over the interior.

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^\pi \int_0^1 r^5(\cos^2\theta + \sin^2\theta) dr d\theta = \pi/6$$