Quiz 0 taken last Friday or tomorrow
Quiz 1 taken next Friday or Tuesday
Homework 3 posted, Worksheets posted
Lecture 3 notes posted

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Off. Hrs: today: 1pm, ACS 362C

Section 12.3: Dot product of vectors (also called scalar or inner product)

We want to take 2 vectors, $\vec{v}$, $\vec{w}$, and combine them to get a scalar (number):

$$\vec{v} \cdot \vec{w} = \text{ a number}$$

If you know $\vec{v}$ and $\vec{w}$ in Cartesian coordinates, then we have a formula:
$\vec{v} = < v_x, v_y, v_z >$
$\vec{w} = < w_x, w_y, w_z >$
$\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z$
easy eh?

Example: $\vec{v} = < 1, e^t, e^{-t} >$ and $\vec{w} = e^t \vec{i} + 3\vec{j} + e^t \vec{k}$
$\vec{v} \cdot \vec{w} = e^t + 3e^t + 1 = 1 + 4e^t$.

The real question is what is the point of doing this? We need to know a bit more before we can answer that.

Basic properties:
$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (commutativity)
$(s\vec{a}) \cdot \vec{b} = \vec{a} \cdot (s\vec{b}) = s(\vec{a} \cdot \vec{b})$ (associativity)
$\vec{a} \cdot \vec{a} = |\vec{a}|^2 =$ length of $\vec{a}$ squared.
$(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$ (distribituvity)

Geometric properties
$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$ with θ the angle between the two vectors.

Proof of scalar product formula

Proof: Consider the triangle with sides $\vec{a}$, $\vec{b}$ and $\vec{a} - \vec{b}$.
What is $|\vec{a} - \vec{b}|^2$?
First way:

$$\begin{aligned} |\vec{a} - \vec{b}|^2 &= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) && (1) \\ &= \vec{a} \cdot \vec{a} - 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} && (2) \\ &= |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a} \cdot \vec{b} && (3) \end{aligned}$$

Now let $\vec{b}_1$ be the projection of $\vec{a}$ onto $\vec{b}$. We then have $|\vec{b_1}| = |\cos\theta||\vec{a}|$. We also define $\vec{b_2} = \vec{b} - \vec{b_1}$ and $\vec{h}$ be the height vector of the triangle. then a second way to compute $|\vec{a} - \vec{b}|^2$ is

$$\begin{aligned}
|\vec{a} - \vec{b}|^2 &= |\vec{h}|^2 + |\vec{b}_2|^2 \\
&= |\vec{a}|^2 \sin^2\theta + (|\vec{b}| - |\vec{b_1}|)^2 \\
&= |\vec{a}|^2 \sin^2\theta + |\vec{b}|^2 - 2|\vec{b}||\vec{a}|\cos\theta + |\vec{a}|^2\cos^2\theta \\
&= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{b}||\vec{a}|\cos\theta
\end{aligned}$$

So then we must have that $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

So the dot product measures angles too: $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$.

Example: What is the angle between $\vec{a} =< 4, 1, 2 >$ and $\vec{b} =< 3, 0, -1 >$?

$$\cos\theta = \frac{10}{\sqrt{21}\sqrt{10}} = \sqrt{\frac{10}{21}}$$

so $\theta = \arccos(\sqrt{10/21})$.

Very important corollary: $\vec{v}$ and $\vec{w}$ are orthogonal ($\perp$) if and only if $\vec{v} \cdot \vec{w} = 0$.

If $\vec{a} =< 1, t, t, >$ and $\vec{b} =< 3, 1, -2 >$. When is $\vec{a} \perp \vec{b}$?
$\vec{a} \cdot \vec{b} = 3 + t - 2t = 3 - t$
so $\vec{a} \cdot \vec{b} = 0$ if $3 - t = 0$ so if $t = 3$ and $\vec{a} =< 1, 3, 3 >$.

Let's return to the idea of a projection:

Projection of $\vec{a}$ onto $\vec{b}$.

The vector projection of $\vec{a}$ onto $\vec{b}$ is the portion of $\vec{a}$ that is in the direction of $\vec{b}$. In other words, it is the "shadow" of $\vec{a}$ if the sun is perpendicular to $\vec{b}$. We denote it as: $\mathrm{Proj}_{\vec{b}}\,\vec{a}$.

The length of the vector projection is called the scalar projection and denoted as: $\mathrm{comp}_{\vec{b}}\,\vec{a}$ = | $\mathrm{Proj}_{\vec{b}}\,\vec{a}$ |.

What is $|\vec{b}_1|$ = $\mathrm{comp}_{\vec{b}}\,\vec{a}$ = | $\mathrm{Proj}_{\vec{b}}\,\vec{a}$ |? $\mathrm{comp}_{\vec{b}}\,\vec{a}$ = $|\vec{a}|\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

So then , how do we compute $\mathrm{Proj}_{\vec{b}}\,\vec{a}$?
First, how do we obtain a vector parallel to $\vec{b}$ of a prescribed length?
We use unit vectors: $\vec{u} = \frac{\vec{b}}{|\vec{b}|}$ has length 1, in the direction of $\vec{b}$.
So

$$\mathrm{Proj}_{\vec{b}}\,\vec{a} = \vec{u}\,\mathrm{comp}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\frac{\vec{b}}{|\vec{b}|} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}.$$

One example of projection you might have seen before is the work in physics, which is the force dotted with the displacement.

Example: $\vec{a} =< 1, 2 >$ and $\vec{b} =< 3, 1 >$

2D Projection example

$\text{Proj}_{\vec{b}}\, \vec{a} = \frac{3+2}{10} < 3, 1 >=< 3/2, 1/2 >$.

This works in 3D just as well, but it is harder to draw: Example: $\vec{a} =< 2, -1, 2 >$ and $\vec{b} =< 0, 3, 1 >$

3D Projection example

$\text{Proj}_{\vec{b}}\, \vec{a} = \frac{0-3+2}{10} < 0, 3, 1 >=< 0, -3/10, -1/10 >$.

What happens if you want to project a vector onto a vector that is perpendicular to the first one?