Lecture 28-29, Section 6.3: Fundamental theorem of Line Integrals

Exercise: Consider $\phi(x,y)$ a scalar function and $\vec{r}(t) =< x(t), y(t) >$ a path.
Along that path, what is ϕ? It is $\phi(x(t), y(t))$.
What is $\frac{d\phi}{dt}$?

Quiz 9 was Friday or will be Tuesday

$$\frac{d\phi}{dt} = \frac{d\phi}{dx}\frac{dx}{dt} + \frac{d\phi}{dy}\frac{dy}{dt} = \nabla\phi \cdot \frac{d\vec{r}}{dt}$$

Quiz 10 is Friday or next Tuesday -> Last one before exam

So what is $\int_a^b \nabla\phi \cdot \frac{d\vec{r}}{dt} dt$?

Notes 28-29 today (fixed typo): Fundamental theorem of Line Integrals
Homework 28-29 posted

$$\int_a^b \nabla\phi \cdot d\vec{r} = \int_a^b \frac{d}{dt}\left(\phi(x(t), y(t))\right) dt = \phi(x(b), y(b)) - \phi(x(a), y(a))$$

So if we call $\vec{F} = \nabla\phi$, $\int_C \vec{F} \cdot d\vec{r} = \phi(Q) - \phi(P)$.

Office hours: today, François, 1-2pm, ACS 362B

Posted:
Old test 2
list of sections covered in test 2
Cheat sheet

The work along ANY path going from P to Q, for a conservative vector field.)

Important points:
1) This only works if $\vec{F} = \nabla\phi$ for some potential $\phi(x,y)$. That is it only works if $\vec{F}$ is CONSERVATIVE.
2) If C is a closed curve, $\oint \nabla\phi \cdot d\vec{r} = \phi(P) - \phi(P) = 0$
3) $\int_C \nabla\phi \cdot d\vec{r}$ does NOT depend on the path taken to go from P to Q. It is called path-independent.
4) $\phi(x,y) = \int_{(0,0)}^{(x,y)} \vec{F} \cdot d\vec{r}$ is a potential to any conservative field $\vec{F}$.

Example: Consider $\phi(x,y) = x \sin y$, then $\nabla\phi = \vec{F} =< \sin y, x \cos y >$
What is the work done by $\vec{F}$ on a particle traveling along a spiral starting at $(0,0)$ and ending at $(1, \pi/2)$?

A spiral (or other complicated path from P to Q.)

$$W = \int_C \vec{F} \cdot d\vec{r} = \phi(1, \pi/2) - \phi(0,0) = \sin(\pi/3) - 0 = \frac{\sqrt{3}}{2}$$

Given $\vec{F} =< F_1, F_2 >$, it would be really good to know:
1) Is $\vec{F}$ conservative?
2) If it is conservative, what is the potential $\phi(x,y)$?

Start with 1). Suppose $\vec{F}$ is conservative and continuous, then there exists $\phi(x,y)$ such that $\nabla\phi = \vec{F}$ and $\frac{\partial\phi}{\partial x} = F_1$ and $\frac{\partial\phi}{\partial y} = F_2$.

Then $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial F_1}{\partial y}$ and $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial F_2}{\partial x}$. If both partial derivatives are continuous, we must have

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

It turns out that this works the other way too: if $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, then $\vec{F} =< F_1, F_2 >$ is conservative.

We call $\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}$ the VORTICITY of $\vec{F}$. If the vorticity is 0, $\vec{F}$ is conservative.

Now for 2), how would we find $\phi(x,y)$ to have $\nabla\phi = \vec{F}$?
We want $\frac{\partial\phi}{\partial x} = F_1$ and $\frac{\partial\phi}{\partial y} = F_2$
So $\phi(x,y) = \int F_1(x,y)dx + C_1(y)$ and
$\phi(x,y) = \int F_2(x,y)dy + C_2(x)$
This is enough to find $\phi(x,y)$, up to a constant c.

Example $\phi(x,y) = x^2e^y + x^3 + 3y + 6$, but we don't know that (supposedly).
Then we have $\vec{F} =< 2xe^y + 3x^2, x^2e^y + 3 >$. The vorticity is $2xe^y - 2xe^y = 0$.

$$\phi(x,y) = \int F_1 dx = \int (2xe^y + 3x^2)dx = x^2e^y + x^3 + C_1(y)$$

and

$$\phi(x,y) = \int F_2 dy = \int (x^2e^y + 3)dy = x^2e^y + 3y + C_2(x)$$

So $C_1(y) = 3y$ and $C_2(x) = x^3$ and $\phi(x,y) = x^2e^y + x^3 + 3y$

What happens in 3D?

Take a potential, $\phi(x,y,z)$ a vector field $\vec{F} =< F_1, F_2, F_3 >= \nabla\phi =< \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} >$. So

$$\frac{\partial\phi}{\partial x} = F_1, \ \ \frac{\partial\phi}{\partial y} = F_2, \ \ \frac{\partial\phi}{\partial z} = F_3,$$

So to have a conservative field, there will now be 3 conditions:

$$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y},$$

and

$$\frac{\partial^2\phi}{\partial x\partial z} = \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x},$$

and

$$\frac{\partial^2\phi}{\partial y\partial z} = \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z},$$

Try $\vec{F}(x,y,z) =< xy, z + x^2/2 + z^2 + xy >$.
Here $(F_1)_z = 0$ and $(F_3)_x = y$, so the field is not conservative.

Now try $\phi(x, y, z) = e^{xz} + x \log y + y^2 + xyz$. Then

$$\vec{F} =< ze^{xz} + \log y + yz, x/y + 2y + xz, xe^{xz} + xy >$$

And $(F_1)_y = 1/y + z$ and so is $(F_2)_x$.
$(F_1)_z = e^{xz}(1 + xz) + y$ and so is $(F_3)_x$.
and $(F_2)_z = x$ and so is $(F_3)_y = x$.
So $\vec{F}$ is conservative, and we can find its potential:

$$\phi(x, y, z) = \int \frac{\partial \phi}{\partial x} dx = \int F_1 dx = \int (ze^{xz} + \log y + yz) dx = e^{xz} + x \log y + xyz + C_1(y, z)$$

And also

$$\phi(x, y, z) = \int \frac{\partial \phi}{\partial y} dy = \int F_2 dy = \int (x/y + 2y + xz) dy = x \log y + y^2 + xyz + C_2(x, z)$$

$$\phi(x, y, z) = \int \frac{\partial \phi}{\partial z} dz = \int F_3 dy = \int (xe^{xz} + xy) dz = e^{xz} + xyz + C_3(x, z)$$

So $\phi(x, y, z) = e^{xz} + x \log y + xyz + y^2 + C$

And $\int_P^Q \vec{F} \cdot d\vec{r} = \int_P^Q F_1 dx + F_2 dy + F_3 dz = \phi(Q) - \phi(P)$ and $\oint \vec{F} \cdot d\vec{r} = 0$.