Lecture 27, Section 6.2: Line Integrals

We are back in 1D domains:
Idea: Your domain is a curve (2D or 3D)

Let me know if you have a conflict with the final on W, May 13, 8am

Quiz 9, taken Friday or next Tuesday

Exam 2 is on April 24
Also posted:
Old Exam 2
List of topics
Cheat Sheet

Today: Lecture notes 27
Homework 27

Office hours today: Hannah, ACS-312, 1:30-3:30pm

The domain of integration of a line integral is a curve.

Today: Line Integrals and work

$C = \vec{r}(t) =< x(t), y(t) >$. Over this curve and maybe more, a function is defined: $f(x, y)$.

Riemann sum version: $\sum_{i=1}^{n} f(x_i, y_i)\Delta s_i$

Riemann sum using a curve as a domain.

where Δs_i is arclength between consecutive points.

$\int_{\vec{r}(t)} f(x,y)ds = \int_C f(x,y)ds = \sum_{i=1}^{n} f(x_i, y_i)\Delta s_i$

Example: 1) Say $f(x, y)$ is a density of raised money and C is the trajectory of a political candidate. Then $\int_C f(x,y)ds$ = total money amassed.
2) $f(x, y)$ is the density of "star" in a video game, in star per length, and C is the trajectory of your "player". $\int_C f(x,y)ds$ = total numer of stars amassed.

Right. Given a trajectory, and a function to integrate, how do we calculate $\int_C f(x,y)ds$?
There are 3 steps:
1) Parametrize C with $\vec{r}(t)$.
2) Rewrite $f(x, y)$ as $f(x(t), y(t))$, a function of t only.
3) Recall $ds = \sqrt{x'(t)^2 + y'(t)^2}dt$.

Example:
$\vec{r}(t) =< t, t^2 >$, for $0 \leq t \leq 2$, so $ds = \sqrt{1+4t^2}dt$

Example 1: a parabola

with $f(x,y) = y/x$, so $f(x(t), y(t)) = t^2/t = t$.

$$\int_C \frac{y}{x} ds = \int_0^2 t\sqrt{1+4t^2}dt = \frac{1}{8}\left.\frac{(1+4t^2)^{3/2}}{3/2}\right|_0^2 = \frac{1}{12}((1+16)^{3/2}-1) = \frac{1}{12}(17^{3/2}-1)$$

Second example: The domain is a circle of radius 3 centered at the origin $\vec{r}(t) =< 3\cos t, 3\sin t >$ so $ds =$

Example 2: a circle

$\sqrt{9\cos^2 t + 9\sin^2 t}dt = 3dt$
$f(x,y) = 15 + y^2$ is the fuel consumption rate. Then the total consumption is

$$\int_C f(x,y)ds = \int_0^{2\pi}(15 + 9\sin^2 t)(3dt) = (15 \cdot 2 \cdot 2\pi + 9\pi)3 = 117\pi$$

Third example: This is an old one: a horizontal line segment, along the x-axis: $\vec{r}(t) =< t, 0 > ds = dt$

Example 3: a horizontal line!

$f(x,y) = g(x)$.

$$\int_C f(x,y)ds = \int_a^b g(t)dt$$

Our good old integral!

MPORTANT SPECIAL CASE: We want to integrate

$$\vec{F}(x,y)\cdot\frac{d\vec{r}(t)/dt}{|d\vec{r}(t)/dt|}$$

What is that?
Well $\vec{F}(x,y)$ is a vector field.
$d\vec{r}/dt$ is a tangent vector to the curve $\vec{r}(t)$.
$\vec{r}'/|\vec{r}'|$ is a UNIT tangent vector, which we also write as $\vec{T}(t)$.

So we have

$$\vec{F}(x,y)\cdot\vec{T} = \text{Proj}_{\vec{T}}\vec{F}$$

which is the portion of $\vec{F}$ in the direction of $\vec{T}$.

Integrating the componenf of $\vec{F}$ tangent to the curve $\vec{r}(t)$.

What does it mean? If $\vec{F}$ is a velocity field, then
$\vec{F}\cdot\vec{T}$ is the velocity of an object moving on C and
$\oint_C \vec{F}\cdot\vec{T}ds$ is called the circulation if C is closed

The Circulation is the integral of the tangent portion of $\vec{F}$ over a closed curve.

It measures how fast things are going around the contour C.

If $\vec{F}$ is a force, $\vec{F}\cdot\vec{T}$ is the portion of the force in the direction of $\vec{T}$, the part that can be used to move things along C.
$\int_C \vec{F}\cdot\vec{T}ds$ = Work done to move an object along C.

Now, given C and $\vec{F}(x,y)$, how do we compute $\int_C \vec{F}\cdot\frac{\vec{r}'(t)}{|\vec{r}'(t)|}ds$? This is also written as $\int_C \vec{F}\cdot d\vec{r}$.

1) Parametrize C using $\vec{r}(t) =< x(t), y(t) >$
2) Note that

$$\vec{T}ds = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}|\vec{r}'(t)|dt = \frac{d\vec{r}}{dt}dt$$

3) Compute $\vec{F}(x,y)$ on C as $\vec{F}(x(t),y(t)) =< F_1(x(t),y(t)), F_2(x(t),y(t)) >$.
4)Take $\vec{F}\cdot\vec{T}ds$ as $F_1(x(t),y(t))\frac{dx}{dt}dt + F_2(x(t),y(t))\frac{dy}{dt}dt$
note that $\frac{dx}{dt}dt$ is sometimes written as dx and $\frac{dy}{dt}dt$ is sometimes written as dy.
5) Integrate the result

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b (F_1(x(t),y(t))\frac{dx}{dt} + F_2(x(t),y(t))\frac{dy}{dt})dt$$

Example: Gravity force $\vec{F}(x,y) =< \frac{-x}{(x^2+y^2)^{3/2}}, \frac{-y}{(x^2+y^2)^{3/2}} >$ Let C be a circle centered at $(0,0)$ of radius 2:

Gravitational force, integrated over a half-circle.

$\vec{r}(t) =< 2\cos t, 2\sin t >$. We will take the upper half only, so $0 \le t \le \pi$. Then we have

$$\vec{F}(x(t),y(t)) =< \frac{-2\cos t}{8}, \frac{-2\sin t}{8} >= -1/4 < \cos t, \sin t >$$

$$\frac{d\vec{r}}{dt} = 2 < -\sin t, \cos t >$$

So the Work is $\int_0^\pi -1/2(-\sin t\cos t + \sin t\cos t)dt = 0$.
Here the force is perpendicular to the displacement.

Example: C is the line from (0,-1) to (2,0), $\vec{r}(t) =< t, -1 + t/2 >$ with $0 \leq t \leq 2$.

A line segment as a domain over which to find the work.

$$\vec{F}(x(t), y(t)) =< -\frac{t}{(5t^2/4 - t + 1)^{3/2}}, \frac{1 - t/2}{(5t^2/4 - t + 1)^{3/2}} >$$

and $\vec{r}'(t) =< 1, 1/2 >$. Then work is then

$$W = \int_0^2 \frac{-t + 1/2 - t/4}{(5t^2/4 - t + 1)^{3/2}} dt = \int_0^2 \frac{1/2 - 5t/4}{(5t^2/4 - t + 1)^{3/2}} dt$$

Let $u = 5t^2/4 - t + 1$ and $du = (5t/2 - 1)dt$.

$$\int_1^4 -1/2 \frac{du}{u^{3/2}} du = \frac{-1}{2} \frac{u^{-1/2}}{-1/2} |_1^4 = -1/2$$

Example $\vec{F}(x, y) =< -y, x >$, solid body rotation.
C is the circle of radius 1 centered at 0 $\vec{r}(t) =< \cos t, \sin t >$.

$$\int_0^{2\pi} < -\sin t, \cos t > \cdot < -\sin t, \cos t > dt = \int_0^{2\pi} 1 dt = 2\pi = \text{ Circulation}$$

If $\vec{F}$ is the velocity of vehicles, $\oint_C \vec{F} \cdot d\vec{r}$ counts cars that go by.