Lecture 25, Section 5.5: Triple Integrals in spherical coordinates

We still want to integrate over a Volume in space, but now we want to describe space using Spherical coordinates: ρ, θ, ϕ.

Quiz 8 is today or Tuesday
Quiz 9 is posted
Notes 25 today: Spherical coordinates
Homework 25 posted

Office hours: today, Lucas, 1-3pm, ACS 362B

Posted:
Old test 2
list of sections covered in test 2
Cheat sheet

Recall spherical coordinates $0 \leq \theta \leq 2\pi$: Angle to the xz-plane

Spherical coordinates.

$0 \leq \phi \leq 2\pi$: Angle to the z-axis
$0 \leq \rho \leq 2\infty$: Distance to the origin

Surfaces easily described in spherical coordinates:
$\rho = R$ is a sphere of radius R, centered at the origin.
$\theta = \pi/4$ is a half-plane.
$\phi = \pi/4$ is a cone, the same as $z = r$ in cylindrical coordinates.

Fixing ρ, θ, and ϕ in spherical coordinates.

The relation between Cartesian and spherical coordinates.
$\rho^2 = x^2 + y^2 + z^2, \quad \cos\phi = z/\rho, \quad \tan\theta = y/x$
$x = \rho\cos\theta\sin\phi, \quad y = \rho\sin\theta\sin\phi, \quad z = \rho\cos\phi.$

To use this in triple integrals $\int\int\int_V f(x,y,z)dV$, we need to:
1) Convert the integrand $f(x,y,z)$ to $f(\rho,\theta,\phi)$.
2) Express V in spherical coordinates.
3) Express dV in spherical coordinates.

So what is dV? $dV =$ length $\times$ width $\times$ height, so
$dV = (\rho\sin\phi)(\rho d\phi)d\rho$
$dV = \rho^2\sin\phi d\theta d\phi d\rho.$

Volume element in spherical coordinates.

Just the spherical volume element.

Example: What is the mass of the Earth if the density is

$$d(x,y,z) = \log(1 - \frac{1}{5}\left(\frac{x^2+y^2+z^2}{R^2}\right)^{3/2})$$

with $R = 6500$ km.

$$\begin{aligned} M &= \int\int\int_V d(x,y,z)dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \log(1-\frac{1}{5}\frac{\rho^3}{R^3}\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta & (1) \\ &= (2\pi)(-\cos\phi\Big|_0^{\pi})(-5/3R^3(\log(1-\frac{1}{5}\frac{\rho^3}{R^3}-1)(1-\frac{1}{5}\frac{\rho^3}{R^3})\Big|_0^{R} & (2) \end{aligned}$$

where we used $u = 1 - 1/5\rho^3/R^3$ and $du = -3/5\rho^2/R^3 d\rho$.
This simplifies to $4\pi R^3(5/3)(1 + (4/5)(1 - \log(4/5))$.

One other example: Mass of an ice cream cone: $\phi = \pi/6$, so the cone is $z = \sqrt{3}(x^2+y^2)^{1/2}$
The ice cream is a spherical cap of radius $R = 2/\sqrt{3}$ centered at $(0,0,R)$.
So the sphere is $x^2 + y^2 + (z-R)^2 = R^2$ or $\rho^2 = 2\rho R\cos\phi$. And finally the density is $d(x,y,z) = \frac{4zy^2}{x^2+y^2}$
So our integral becomes:

$$\int_0^{2\pi}\int_0^{pi/6}\int_0^{2R\cos\phi} \frac{4\rho\cos\phi\rho^2\sin^2\phi\sin^2\theta}{\rho^2\sin^2\phi}\rho^2\sin\phi d\rho d\phi d\theta$$

Ice cream cone.