Lecture 24, Section 5.5: Triple Integrals in cylindrical coordinates

We still want to integrate over a Volume in space, but now we want to describe space using Cylindrical coordinates: r, θ, z.

Recall cylindrical coordinates

Quiz 7, Friday before break or Tuesday
Quiz 8, taken Friday or next Tuesday

Today: Lecture notes 24
Homework 24

Office hours today: François, ACS-362, 1-2pm

Today: Triple integral example and cylindrical coordinates

Cylindrical coordinates.

r and θ are the same as in polar coordinates, and z is the same as in Cartesian coordinates.
r is the distance to the z-plane, $r^2 = x^2 + y^2$
$0 \leq \theta \leq 2\pi$: Angle to the xz-plane, so $x = r\cos\theta, y = r\sin\theta$
z is the vertical height.

Surfaces easily described in cylindrical coordinates:
$z = K$ is a horizontal plane of height K.
$r = R$ is a vertical cylinder
$\theta = \pi/4$ is a half-plane.

Fixing r, θ, and z in cylindrical coordinates.

To use this in triple integrals $\int\int\int_V f(x, y, z)dV$, we need to:
1) Convert the integrand $f(x, y, z)$ to $f(r, \theta, z)$.
2) Express V in cylindrical coordinates.
3) Express dV in cylindrical coordinates.

So what is dV? $dV =$ length X width X height, so

Volume element in cylindrical coordinates.

$dV = rd\theta drdz$

Example: What is the volume of the cone $z = 2\sqrt{x^2+y^2}$ below the height $z = H$?

$$\begin{aligned} V &= \int\int\int_V d(x,y,z)dV = \int_0^{2\pi}\int_0^{H/2}\int_{2r}^{H} dz\ r\ dr\ d\theta \\ &= (2\pi)\int_0^{H/2}(H-2r)rdrd\theta = (2\pi)(H(r^2/2) - 2r^3/3) = 2\pi(H^3/8 - 2H^3/24) = \pi H^3/12 \end{aligned}$$

Note that this is the same as $V = \pi R^2 H/3$, because here $R = H/2$.

Cone $z = 2\sqrt{x^2+y^2} = 2r$ and paraboloid $z = 2(x^2+y^2) = 2r^2$

What changes if we want the volume inside the paraboloid $z = 2(x^2+y^2)$? Our integral is then:

$$V = \int\int\int_V d(x,y,z)dV = \int_0^{2\pi}\int_0^{\sqrt{H/2}}\int_{2r^2}^{H} dz\ r\ dr\ d\theta$$

Example: what is the average height of points inside the upper half-sphere of radius R? (This is the vertical coordinate of the center of mass).
Here the surface bounding the volume above is $x^2+y^2+z^2 = R^2$ or $z = \sqrt{R^2-(x^2+y^2)}$ or $z = \sqrt{R^2-r^2}$.

So we want:

$$
\begin{aligned}
\bar{z} &= \int\int\int_V z dV = \int_0^{2\pi}\int_0^R\int_0^{(R^2-r^2)^{1/2}} z\ dz\ r\ dr\ d\theta \\
&= (2\pi)\int_0^R (R^2-r^2) r dr = 2\pi(R^2r^2/2 - r^4/4)\,|_{r=0}^R = \pi R^4/2
\end{aligned}
$$

upper half-sphere of radius R and sphere of radius R

Finally, what is the volume of a sphere of radius R?

$$
\begin{aligned}
V &= \int\int\int_V dV = \int_0^{2\pi}\int_0^R\int_{-(R^2-r^2)^{1/2}}^{(R^2-r^2)^{1/2}} dz\ r\ dr\ d\theta \\
&= (2\pi)\int_0^R 2(R^2-r^2)^{1/2} r\ dr = 2\pi(2/3)(1/2)(R^2-r^2)^{3/2}\,|_{r=0}^R = 4\pi R^3/3
\end{aligned}
$$

Fantastic, no?