Lecture 22, Section 5.4: Triple Integrals

Going from double to triple integral is a ”natural” extension.

We still have Riemann sum definition:

$$\int\int\int_V f(x,y,z)dV = \lim_{m,n,p\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(x_i,y_j,z_k)\Delta V$$

where ΔV is the volume element.

Break space up into small rectangular prism (boxes). Our domain in now in 3D.

Breaking up 3D space into small boxes.

Our integrand is now a function of 3 variables.

What does this all mean?
2 common uses:
1) $\int\int\int_V dV =$ Volume of V.
all double integrals give : $\int\int_D f(x,y)dA = \int\int_D \left(\int_0^{f(x,y)} dz\right) dA$

Volume computed by adding areas.

2) If $f(x,y,z)$ is a density, $\int\int\int_V f(x,y,z)dV$ is a mass.

Recall: density = mass / volume.

This can be made into a local statement: density changes in space.

What is the mass of water over Merced County?
Let $\rho(x,y,z)$ be the density of water in the air, in kilograms per meter cubed.

Mass computed via a triple integral.

$$M = \int\int_A \int_0^\infty \rho(x,y,z) dz dA$$

blondie

We still have to shoot arrows, but now in one more direction. Example calories in a brownie.

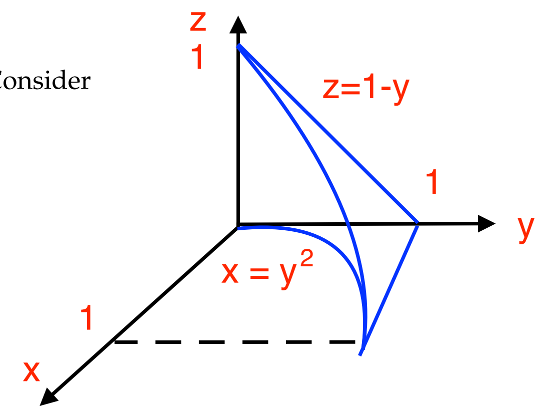
$$\int\int\int_V (3 + xz - y^3) dV$$

with $V = \{(x,y,z) | -1 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 1\}$.
First, you must sketch V. This is important! For a box, we can start in any direction: You try.

A first triple integral.

$$\int_{-1}^{1} dx \int_0^2 dy \int_0^1 dz (xz - y^3) = \int_{-1}^{1} dx \int_0^2 dy (x/2 - y^3) = \int_{-1}^{1} dx (x - 4) = -8 + 12 = 4$$

More challenging is the following domain. Consider



A more complex domain.

$$I = \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)dzdydx$$

What is V? (figure) Same as $\int_0^1 \int_0^{y^2} \int_0^{1-y} f(x,y,z)dzdxdy$

Or start it with y (figure)

$$I = \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)dydxdz$$

or

$$I = \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)dydzdx$$

Or start it with x (figure)

$$I = \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)dxdydz$$

or

$$I = \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)dxdzdy$$