Quiz 7, taken Friday or Tuesday after break

Lecture notes 22
Homework 22 posted

Lecture 22, Section 5.3: Integration in polar coordinates

Office hours: Hannah, today, 1:30-3:30pm ACS-312

Recall double integrals:
The volume between $z = 0$ and $z = f(x, y)$ over D is $V = \int\int_D f(x, y)dA$, a number
The area of a region in the xy-plane is $A = \int\int_D dA$
The average of a function over a domain D is

Today: Integration in polar coordinates (5.3)

$$\bar{f} = \frac{\int\int_D f(x, y)dA}{\int\int_D dA}$$

Midterm: Portion of Final exam can replace first midterm grade

with the following properties:

$$\int\int_D (af(x, y) + bg(x, y))dA = a\int\int_D f(x, y)dA + b\int\int_D g(x, y)dA$$

$$\int\int_{D_1} f(x, y)dA + \int\int_{D_2} f(x, y)dA = \int\int_{D_1+D_2} f(x, y)dA$$

and with definition

$$\int\int_D f(x, y)dA = \lim_{n\to\infty, m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j)\Delta x\Delta y$$

with $\Delta x\Delta y = dA$.

This definition is equivalent of splitting the domain D into rectangles, and adding them up.

Splitting a domain in cartesian pieces

But there are other ways to split a domain into small pieces

Two more ways to split a domain.

You can even use polar coordinates!
Every point (x, y) may be written as $(r\cos\theta, r\sin\theta)$, or in polar form (r, θ).
In the other direction, we have $r = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$.

In polar form, $f(x, y)$ may be rewritten $f(r\cos\theta, r\sin\theta)$. So we can rewrite the integral as

$$\int\int_D f(x,y)dA = \int\int_D f(r\cos\theta, r\sin\theta)dA = \sum_{i=1}^{n}\sum_{j=1}^{m} f(r_i, \theta_j)dA$$

We can now try to describe D and dA in terms of r and θ.

But what is dA?

The area element in polar coordinates.

$$dA = \frac{\pi(r+\Delta r)^2 - \pi r^2}{2\pi}\Delta\theta = r\Delta r\Delta\theta + (\Delta r)^2\Delta\theta/2$$

As we take $\Delta r \to 0$, the last term is much smaller than the others and so it is negligible.
You may think of this area as a bent rectangle of size Δr by $\Delta\theta r$.

So we will use $dA = r\ dr\ d\theta$ in our integrals. This has the units of area, which can help you remember it.

For the bounds of integration, you still shoot "arrows" (automatic weapon/boomerang)

Arrows (Cartesian) vs boomerang (polar)

Example 1: a half-circle of radius 2

What is the average of $f(x,y) = \sin(x^2+y^2)$ over D?

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\int_0^2 \sin(r^2) r dr d\theta = \left(\frac{1}{2\pi}\pi\right)\left(-\frac{\cos(r^2)}{2}\bigg|_0^2\right) = \frac{1-\cos 4}{4}$$

Example 2: Let $f(x,y) = xy$. Domain is $(x-1)^2 + y^2 \leq 1$. We rewrite $(x-1)^2+y^2 = 1$ as $x^2+y^2 = 2x$ so

Example 2: a translated circle of radius 1

$r^2 = 2r\cos\theta$ so $r = 2\cos\theta$.

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3 \sin\theta\cos\theta dr d\theta = \int_{-\pi/2}^{\pi/2} 16\cos^5\theta\sin\theta d\theta$$

And the rest is (supposed to be by now) easy.

Example 3

What is the area of the region bounded by $r = \cos(2\theta)$, for $-\pi/4 \leq \theta \leq \pi/4$.

$$A = \int_{-\pi/4}^{\pi/4} \int_0^{\cos(2\theta)} dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos(2\theta)} r dr d\theta$$

$$\int_{-\pi/4}^{\pi/4} \frac{\cos^2 \theta}{2} d\theta = \frac{2\theta + \sin 4\theta}{4} |_{-pi/4}^{\pi/4} = \pi/4$$

Their are 2 reasons to use polar coordinates rather than Cartesian coordinates:
1) The integrand is suitable to polar coordinates
OR
2) The domain is suitable to polar coordinates.