Lectures 20, 21, Section 5.2: Integrating over general domains

In this section, we need to be able to:
1) Integrate over a given domain.
2) Find a domain given bounds of integration.

Given an integral, we can find the corresponding domain of integration

$$\int_{x=a}^{x=b} dx \int_{y=f(x)}^{y=g(x)} dy\ h(x,y)$$

Recall that the outer bound must be constant.
The inner bound can depend on the OTHER variable.
Here, the domain of integration is $D = \{a \le x \le b, f(x) \le y \le g(x)\}$. This is called a Type I domain (by the book anyway).

Quiz 6, last Friday or Tuesday
Quiz 7, taken Friday or Tuesday after Spring break

Today: Lecture notes 20-21
Homework 20-21 posted

Office hours canceled today,

Today: Examples of integrating over a general domain

A domain where you should integrate y first.

Similarly, we can have:

$$\int_{y=c}^{y=d} dy \int_{x=m(y)}^{x=n(y)} dx\ h(x,y)$$

with $D = \{c \le y \le d, m(y) \le x \le n(y)\}$, a type II domain.

A domain where you should integrate x first.

Note that we can integrate over a mixture of type I and type II domains using

$$\int\int_{D_1} h(x,y)dA + \int\int_{D_2} h(x,y)dA$$

Coming up with a figure is ESSENTIAL to understanding the domain of integration.

A domain that needs to be broken up.

How can we go from the bounds of integration to the picture of the domain? Given a domain, find the correct bounds of integration. We call on Cupid and his arrows.

First, we shoot an vertical arrow, to find the inner integral: $\int_{in}^{out}$.
For the outer integral, we look for bounds on where an arrow can be shot from: $\int_{left/bottom}^{right/top}$.

Consider the region below $y = 3x - 2$, above $y = (x+1)/2$ and for $1 \leq x \leq 4$.
Here $\int_1^4 dx \int_{x/2+12}^{3x-2} dy h(x, y)$.

Domain corresponding to $\int_1^4 dx \int_{x/2+12}^{3x-2} dy h(x, y)$.

How do we find the area of that triangle? Just use $h(x, y) = 1$, so that all we integrate is the area element. We find

$$\int_1^4 (3x - 2 - x/2 - 1/2)dx = 5x^2/4 - 5x/2\Big|_1^4 = 20 - 10 - 5/4 + 5/2 = 11\ 1/4$$

Example: Right Half-circle of radius 3, centered at the origin

We have 2 choices:
Vertically first:

$$\int_0^3 dx \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\ h(x, y)$$

or horizontally first

$$\int_{-3}^3 dy \int_0^{\sqrt{9-y^2}} dx\ h(x, y)$$

A half-circle.

So if $h(x,y)=x$, we find

$$\int_{-3}^{3} dy \int_{0}^{\sqrt{9-x^2}} dyx = \int_{-3}^{3} x^2/2\Big|_0^{\sqrt{9-y^2}} dy = \int_{-3}^{3} \frac{9-y^2}{2} dy = 9/2y - y^3/6\Big|_{-3}^{3} = 27-9 = 18$$

or similarly in the other direction

$$\int_{0}^{3} dx \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dyx = \int_{0}^{3} dx\; xy|_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} = \int_{0}^{3} dx 2x(\sqrt{9-x^2}) = -(2/3)(9-x^2)^{3/2}\Big|_0^3 = (2/3)27 = 18$$

We can now deal with more complicated domains. Consider the heart shape bounded by:
$y=-2x-4$ on the bottom left,
$y=2x-4$ on the bottom right,
$(x+1)^2+y^2=1$ and $y>0$ on the top left,
$(x-1)^2+y^2=1$ and $y>0$ on the top right.

A more complicated domain.

What to do? Break it!

$$\int\int_D h(x,y)dA = \int\int_{D_1} h(x,y)dA + \int\int_{D_2} h(x,y)dA + \int\int_{D_3} h(x,y)dA + \int\int_{D_4} h(x,y)dA$$

$$= \int_{-2}^{0} dx \int_{-2x-4}^{0} dyh(x,y) + \int_{-4}^{0} dy \int_{0}^{(y+4)/2} dxh(x,y) + \int_{-2}^{0} dx \int_{0}^{\sqrt{1-(x+1)^2}} dyh(x,y) + \int_{0}^{2} dx \int_{0}^{\sqrt{1-(x-1)^2}} dyh(x,y)$$

One more example, where we need to reverse the order of integration.

$$\int_{x=0}^{x=2} \int_{y=x}^{y=2} e^{y^2}\, dy\, dx$$

From the given bounds, draw the domain

Here, the first integral cannot be done using an antiderivative (no elementary function is an antiderivative of e^{y^2}. But we can use a trick: Reverse the order of integration:

Consider the domain we drew, what would be the bounds of integration if we started in the x-direction?

Redraw the domain, but start in the x-direction

$$\int_{y=0}^{y=2} \int_{x=0}^{x=y} e^{y^2}\, dx\, dy$$

Now the first integral is simple:

$$\begin{aligned}
\int_{y=0}^{y=2} \int_{x=0}^{x=y} e^{y^2}\, dx\, dy &= \int_{y=0}^{y=2} xe^{y^2}\Big|_{x=0}^{x=y}\, dy \\
&= \int_{y=0}^{y=2} ye^{y^2}\, dy \ \text{ use } u = y^2, du = 2ydy \\
&= \frac{1}{2}e^{y^2}\Big|_{y=0}^{y=2} \\
&= \frac{e^4 - 1}{2}
\end{aligned}$$