Section 2.2: Vectors

If you connect two points in space, with a specific direction (from a starting point to a finishing point), you get a VECTOR. You can think of it as an arrow.

A vector

A vector has a given LENGTH and a DIRECTION (but it doesn't really have a single starting point or a single finishing point). A vector can be translated and remains the same vector, but if it is rotated, it becomes a new vector.

A vector will be denoted by its 3 Cartesian components, within $<,,>$,

$$\vec{v} = < x, y, z >$$

where each component stands for the displacement in the x, y, or z direction.

What is the vector connecting $P = (2, 2, 1)$ to $Q = (0, 4, 2)$? We would like $P + \vec{v} = Q$ so

$$\vec{v} = \vec{PQ} = Q - P = (0, 4, 2) - (2, 2, 1) = < -2, 2, 1 >$$

where we take the arrival point and subtract the starting point.

If we let $\vec{v} = < v_1, v_2, v_3 >$ and $\vec{w} = < w_1, w_2, w_3 >$ we can perform arithmetic component-by-component:
If $c \in \mathbb{R}$, $c\vec{v} = < cv_1, cv_2, cv_3 >$
and $\vec{v} + \vec{w} = < v_1 + w_1, v_2 + w_2, v_3 + w_3 >$

There are 3 special vectors that form the Cartesian basis:

$$\vec{i} = < 1, 0, 0 > \qquad \vec{j} = < 0, 1, 0 > \qquad \vec{k} = < 0, 0, 1 >$$

They are perpendicular (or ORTHOGONAL) to each other, and have length one. This is called an orthonormal basis.

$$\vec{i} = < 1, 0, 0 > \qquad \vec{j} = < 0, 1, 0 > \qquad \vec{k} = < 0, 0, 1 >$$

This gives us another way of writing vectors:

$$\vec{v} = < 3, -1, 4 > = 3\vec{i} - \vec{j} + 4\vec{k} = 3 < 1, 0, 0 > - < 0, 1, 0 > + 4 < 0, 0, 1 >$$

Useful properties for vectors denoted $\vec{a}$, $\vec{b}$ and $\vec{c}$ are on in section 2.2 (pages 122, 123) of the textbook You can define a vector space from those properties, but that is done in Math 24.

We will look at the geometrical representation of some operators:

Addition subtraction multiplication by a scalar

What is the length, or magnitude, of a vector?
If you start at the origin, the length is the distance from O to the tip of the vector $\vec{v}$. So we write, for $\vec{v} =< v_1, v_2, v_3 >$,

$$|\vec{v}| = ||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

What is the length of $c\vec{v}$? It is just the (absolute value) of the product: $|c||\vec{v}|$.

So how can you use this to get a vector of length 1, a so-called unit vector, going in the same direction as a given vector $\vec{v}$? You try with $\vec{v} = 2\vec{i} + 7\vec{j} - \vec{k}$.

$$\vec{u} = \text{unit vector parallel to } \vec{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{2\vec{i} + 7\vec{j} - \vec{k}}{\sqrt{54}} =< 2/\sqrt{54}, 7/\sqrt{54}, -1/\sqrt{54} >$$

Finding and using unit vectors is a big deal.

You can also describe a vector by its length and the angle(s) it makes with a given direction: In 2D, you use polar coordinates.
Say $\vec{v}$ has length 5 and makes an angle of $3\pi/4$ with the x-axis, so $\vec{v} =< 5\cos 3\pi/4, 5\sin 3\pi/4 >$.
In general, $\vec{v} =< r\cos\theta, r\sin\theta >$

$$\vec{v} =< 5\cos 3\pi/4, \quad 5\sin 3\pi/4 >$$

In 3D, we would need 2 angles. One is the angle with the xz-plane, we still call it θ, and the other is the angle with the z-axis, we call is ϕ. For example $\vec{w} =< 5\cos\theta\sin\phi, 5\sin\theta\sin\phi, 5\cos\phi >$. We will get back to this, the SPHERICAL COORDINATES.

$$\vec{w} =< 5\cos\theta\sin\phi, 5\sin\theta\sin\phi, 5\cos\phi >.$$

Important point: When adding or multiplying, or later doing other operations, we have to make sure to compare comparable things. For example:
$\vec{v} + 3$ has no meaning
$\vec{v}\vec{w}$ has no meaning but $\vec{v} + \vec{w}$ does
$\vec{v} + |\vec{v}|$ has no meaning.