Lecture 19, Section 5.2: Iterated integrals

Recall: Signed Volume between $z = 0$ and $z = f(x, y)$ is

$$V = \int_a^b dx \int_c^d dy\, f(x,y) = \lim_{m\to\infty, n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x^*, y^*)(x_i - x_{i-1})(y_j - y_{j-1}) = \int_a^b dx \int_c^d dy f(x,y)$$

Idea: Add up $f(x, y)$ multiplied by the size of the region over which it is applied.

How do you compute such a signed volume?

Idea: Add up slices. What is the volume of a slice? It is the area under the curve, times Δx.

Quiz 6, taken Friday or Tuesday

Lecture notes 19
Homework 19 posted

Office hours: Hannah, today, 1:30-3:30pm ACS-312

Today: Iterated integrals (5.2)

Slicing a volume with fixed x.

$V = \lim_{n\to\infty} \sum_{i=1}^{n}$ area under the curve $(x_i - x_{i-1})$.
This is the same as

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\int_c^d f(x^*, y) dy \right) (x_i - x_{i-1})$$

where each integral is taken for some fixed x^*. As a result, we get an area for each x^*.

The limit of the sum that is left is actually an integral over x, so we can complete the computation by taking the integral of the result we just obtained:

$$\int_a^b dx[\int_c^d f(x,y)dy]$$

Similarly, we can start with x $V = \lim_{m\to\infty} \sum_{j=1}^{m}$ area under the curve $(y_j - y_{j-1})$.
This is the same as

$$\lim_{m\to\infty} \sum_{j=1}^{m} \left(\int_a^b f(x, y^*) dx \right) (y_j - y_{j-1})$$

where each integral is taken for some fixed y^*. And again, we can finish the problem by taking the integral of the result

$$\int_c^d dy[\int_a^b f(x,y)dx]$$

Either order gives the same answer if the original function is continuous.
We evaluate the integrals one at a time, treating the other variable as constant:

Slicing a volume with fixed y.

A rectangular domain of integration.

Example $D = [0, 2] \times [1, 3]$

$$\int\int_D (x + 2y + 3)dA = \int_0^2 dx \int_1^3 dy(x + 2y + 3)dA = \int_0^2 dx \left(\int_1^3 (x + 2y + 3)dy\right)$$

or alternatively

$$\int_1^3 dy \left(\int_0^2 (x + 2y + 3)dx\right) = \int_1^3 dy x^2/2+2xy+3x \mid_0^2 = \int_1^3 (2+4y+6-0)dy = 8y+2y^2 \mid_1^3 = 24+18-8-2 = 32$$

Fubini's theorem: If $f(x,y)$ is continuous, then iterated integrals can be evaluated in either order.

Sometimes one order is easier than the other (see examples later).

Example

$$\int_1^2 \int_0^1 \frac{xe^x}{y} dxdy = \int_1^2 \frac{dy}{y} \int_0^1 xe^x dx = \log y \,\big|_1^2 (xe^x - e^x\big|_0^1) = \log 2$$

What if the domain of integration is not a rectangle? Use arrows Here $0 \le y \le x/2$ for $0 \le x \le 3$ so we can

Using arrows to determine the bounds of integration.

integrate

$$\int_0^3 dx \int_0^{x/2} dy f(x,y) = \int_0^{3/2} dy \int_{2y}^3 dx f(x,y)$$

Important: These integrals compute a signed volume, so they should give a NUMBER. That means
1) The bounds of the inner integral can depend on the other variable ONLY
2) The bounds of the outer integral HAVE TO be numbers.