Lecture 18, Section 5.1: Double integrals

Recall the Riemann sums:

Riemann sum for a function of one variable

To calculate the area under a curve, use rectangles (because you know their area).

The idea is that, as you take more and more rectangles (smaller and smaller ones), you get a better approximation of the area under the curve.

$$A = \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) \text{ with } x_i^* \in [x_{i-1}, x_i]$$

A few remarkable facts:
1) The limit exists for any continuous functions (and more really)
2) This works for any $x_i^* \in [x_{i-1}, x_i]$.
3) $\int_0^x f(x)dx = F(x)$ is the ANTIDERIVATIVE of $f(x)$. So $F'(x) = f(x)$.

The goal of this class is for you to be able to integrate over any domain:
1) A part of the xy-plane
2) A part of $\mathbb{R}^n$
3) A curve
4) A general surface.

The idea is always the same: Add up all the values of $f(x^*)$, multiplied by the size of the region within which x^* is taken.

So consider $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$
Its domain is in $\mathbb{R}^2$ so we will integrate over a region of $\mathbb{R}^2$, a rectangle for now.

We will approximate the VOLUME under the surface $f(x, y)$ using rectangular prisms (boxes).

2D Domain of integration and Riemann sum for a function of 2 variables.

$$V \approx \sum_{j=1}^{n} \sum_{i=1}^{n} f(x^*, y^*)(x_i - x_{i-1})(y_j - y_{j-1})$$

with $(x^*, y^*) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$.

We will get progressively better estimates as the rectangles get smaller, so that

More boxes yield a more accurate approximation of the integral.

$$V = \int_a^b dx \int_c^d dy f(x, y) = \lim_{m\to\infty, n\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{n} f(x^*, y^*)(x_i - x_{i-1})(y_j - y_{j-1})$$

Remarkably:
1) This still works for any continuous function $f(x, y)$ (and more really).
2) You can still use any (x^*, y^*) in each small rectangle
3) Using antiderivatives, you can evaluate it again (we'll see how).

The best point to use to approximate these integrals is the one in the middle of the rectangle:

$$(x^*, y^*) = \left(\frac{x_i + x_{i-1}}{2}, \frac{y_j + y_{j-1}}{2}\right)$$

So we have a first way to evaluate integrals: add up volumes of all the boxes (computerized is better). We will see other ways next time.

There are other useful interpretations of the integrals:
Average of a function. In 2D, we had

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx = \lim_{n\to\infty} \frac{1}{\sum_{i=1}^n (x_i - x_{i-1})} \sum_{i=1}^n f(x^*)(x_i - x_{i-1}) = \frac{\int_a^b f(x)dx}{\int_a^b dx}$$

Similary, in 3D:

$$\bar{f} = \frac{\int_a^b dx \int_c^d dy f(x,y)}{\int_a^b dx \int_c^d dy} = \frac{\int_a^b dx \int_c^d dy f(x,y)}{(b-a)(d-c)}$$

where the bottom represents the area of the rectangle over which we are integrating.

Note that this will be true even if the area over which we integrate is not a rectangle.

Note that we still have:

$$\int\int_D f(x,y)dxdy + \int\int_D g(x,y)dxdy = \int\int_D (f(x,y) + g(x,y))dxdy$$

$$\int\int_D cf(x,y)dA = c\int\int_D f(x,y)dA$$

$$\int\int_{D_1} f(x,y)dxdy + \int\int_{D_2} f(x,y)dxdy = \int\int_{D_1+D_2} f(x,y)dxdy$$

And finally

$$\int\int_D dA = \text{ Area of D}$$