Lecture 17, Section 4.7: Optimization

Remember local min/max in 2D: requires $f'(x) = 0$

2D optimization: Max and min occur at points where the curve is horizontal.

Similarly in 3D, at a local min/max, the surface is flat. That is to say, the tangent plane is horizontal. So at

3D optimization: Max and min occur at points where the tangent plane is horizontal.

a local min/max (a,b), we have
$f_x(a, b,) = 0$ and $f_y(a, b) = 0$ SIMULTANEOUSLY.
But that isn't quite enough: Remember the pringles?

So how do we tell? And how do we know if we have a min or a max?
Use Taylor Series expansion, to the SECOND order. Recall:

$$f(a + \Delta x) = f(a) + f'(a)\Delta x + f''(a)(\Delta x)^2/2 + O((\Delta x)^3)$$

So in 3D, we get:

$$f(a+\Delta x, b+\Delta y) = f(a, b)+f_x(a, b)\Delta x+f_y(a, b)\Delta y+f_{xx}(a, b)(\Delta x)^2/2+f_{xy}(a, b)(\Delta x\Delta y)+f_{yy}(a, b)(\Delta y)^2/2$$

Why does that help? Because I know:
$f(u, v) = u^2 + v^2 + c$ has a minimum at $(0, 0)$.
$f(u, v) = -u^2 - v^2 + c$ has a maximum at $(0, 0)$.
$f(u, v) = u^2 - v^2 + c$ has a saddle point at $(0, 0)$.

So at a critical point (where $f_x = f_y = 0$), we have:

$$f(a + \Delta x, b + \Delta y) = f(a, b) + f_{xx}(a, b)(\Delta x)^2/2 + f_{xy}(a, b)(\Delta x\Delta y) + f_{yy}(a, b)(\Delta y)^2/2$$

If we complete the square (assuming that $f_{xx} \neq 0$) we get

$$f(a + \Delta x, b + \Delta y) - f(a, b) = 1/2(f_{xx}(\Delta x + f_{xy}/f_{xx}\Delta y)^2 + (\Delta y)^2(f_{yy} - (f_{xy})^2/f_{xx}))$$

and if we let $u = \Delta x + f_{xy}/f_{xx}\Delta y$, $v = \Delta y$ and $D = f_{xx}f_{yy} - f_{xy}^2$, we get

$$f(a+u, b+v) - f(a,b) = 1/2(f_{xx}u^2 + v^2(D/f_{xx}))$$

So if:
$f_{xx} > 0$ and $D = f_{xx}f_{yy} - f_{xy}^2 > 0$, then we have a minimum.
$f_{xx} < 0$ and $D = f_{xx}f_{yy} - f_{xy}^2 > 0$, then we have a maximum.
$D < 0$ we have a saddle point.

If $D = 0$, this is inconclusive. We would then need to look either at higher derivatives, or use other means to figure it out. For example, $f(x,y) = x^6 + y^8$ has a minimum at $(0,0)$ because everywhere else it is positive.

Example:$f(x,y) = x^3/3 - 5x^2/2 + 4x + 100 + (y-2)^2$
Then $f_x = x^2 - 5x + 4 = (x-4)(x-1)$
and $f_y = 2(y-2)$. So we have TWO critical points: (1,2) and (4,2). Let us classify them:
$f_{xx} = 2x - 5$
$f_{xy} = 0$
$f_{yy} = 2$
So $D = 4x - 10$. If $x = 1$, $y = 2$, we have $D = -6 < 0$ so it is a saddle-point at (1,2).
If $x = 4$, $y = 2$, $D = 6 > 0$ and $f_{xx} = 3$ so we have a local minimum at (4,2)

Constrained Optimization: A global max or min is a point (x_0, y_0) such that: $f(x_0, y_0) \geq f(x,y)$ for all points (x,y) in the domain under consideration.

A Global max/min may occur:
1) At a local max/min
2) On a boundary of the domain, including at corners if applicable.
3) Nowhere, if $f(x,y) \to \infty$ or $-\infty$ within the domain (including as x or y approaches infinity.

Example: Let x be the time spent studying math in a day, and y be the time spent studying anything in a day.
Let the accumulated knowledge be $f(x,y) = x^3/3 - 5x^2/2 + 4x + 100 + (y-2)^2$.
We want to optimize this over the region $x \geq 0$, $y \geq x$ and $y \leq 24$.

Domain for constrained optimization.

Is there a local min or max? Well we found critical points (1,2) and (4,2) before. Only (1,2) is within our domain, and it was a saddle-point.

So we need to check along the boundaries.
Along $x = 0$, we have
$f(0,y) = (y-2)^2 + 100$
which has a minimum at $y = 2$, where $f(0,2) = 100$.

Along $y = 24$, we have

$f(x, 24) = g(x) = x^3/3 - 5x^2/2 + 4x + 100 + 484$
. So we find $g'(x) = 0$ if an only if $x = 1$ and $x = 4$. Both are within our domain.
We have $g''(x) = 2x - 5$ so at $x = 1$, we have a local maximum, and at $x = 4$ a local minimum.
We have $f(1, 24) = 1/3 - 5/2 + 4 + 584 = 584 + 11/6$ and
$f(4, 24) = 64/3 - 40 + 16 + 584 = 584 - 8/3$

Along $y = x$, we have
$f(x, x) = g(x) = x^3/3 - 5x^2/2 + 4x + 100 + (x - 2)^2$
. $g'(x) = x^2 - 5x + 4 + 2(x - 2) = x^2 - 3x$
So $g'(x) = 0$ at $x = 0$ and $x = 3$. We also have
$g''(x) = 2x - 3$ so at $x = 0$ we have a maximum and $f(0, 0) = 104$ and at $x = 3$ we have a local minimum and $f(3, 3) = 9 - 45/2 + 12 + 100 + 1 = 99 + 1/2$.

Finally, we need to check the corner of our domain:
$f(0, 0) = 104$ as we know already
$f(0, 24) = 100 + 484 = 584$ and
$f(24, 24) = 24^3/2 - 524^2/2 + 4 * 24 + 100 + 484 = 6152$.

So among all our candidates for global min and max, we see that the maximum knowledge is achieved at $(24, 24)$, and the minimum at $(3, 3)$.

So the best is to study math all the time, but if you don't do that enough you get the worst, by thinking that you know when actually you don't.