Lecture 14, Section 4.6: Directional derivatives

We can now take a derivative in the x-direction (f_x) or y-direction (f_y). How about other directions?

Quiz 5, today Friday or Tuesday

Lecture notes 15-16
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Office hours: Lucas, today, 1:00-3:00pm ACS-362,

Derivative in a general direction (seen on a surface and on a contour plot).

Today: Gradients

Consider a UNIT vector $\vec{u} = u_x\vec{i} + u_y\vec{j} =< u_x, u_y >$, $||\vec{u}|| = 1$.
We define the DIRECTIONAL DERIVATIVE as:

$$D_{\vec{u}}f(a,b) = f_{\vec{u}}(a,b) = \lim_{h\to 0}\frac{f(a+hu_x, b+hu_y) - f(a,b)}{h} = \lim_{h\to 0}\frac{f((a,b)+\vec{u}h) - f(a,b)}{h}$$

Meaning: How fast does f change as you look in the direction $\vec{u}$?

Let us use our linear approximation on $f(a+hu_x, b+hu_y)$:

$$f(a+hu_x, b+hu_y) \approx f(a,b) + hu_xf_x(a,b) + hu_yf_y(a,b) + h\epsilon$$

with ϵ going to 0 as $h \to 0$.

So we get

$$D_{\vec{u}}f(a,b) = \lim_{h\to 0}\frac{hu_xf_x(a,b) + hu_yf_y(a,b) + h\epsilon}{h} = u_xf_x(a,b) + u_yf_y(a,b)$$

Note that this only works if $||\vec{u}|| = 1$.
If the given direction $\vec{v}$ is not unitary, just use $\vec{u} = \frac{\vec{v}}{||\vec{v}||}$ in the formula.

Example: $f(x,y) = x\cos y$, $\vec{v} =< -3, 4 >$, find $f_{\vec{v}}(2, \pi/4) = D_{\vec{v}}(2, \pi/4)$
We get first $\vec{u} =< -3/5, 4/5 >$
and $f_x = \cos y$ so $f_x(2, \pi/4) = \frac{\sqrt{2}}{2}$
and $f_y = -x\sin y$ so $f_y(2, \pi/4) = -\sqrt{2}$
so $D_{\vec{v}}(2, \pi/4) = -3/5\sqrt{2}2 - 4/5\sqrt{2} = \sqrt{2}(-11/10)$

We now introduce an important vector:
The GRADIENT of a function $f(x,y)$ is the VECTOR

$$\text{grad} f(a,b) = \nabla f(a,b) = f_x(a,b)\vec{i} + f_y(a,b)\vec{j} =< f_x(a,b), f_y(a,b) >$$

First use: Rewrite the directional derivative formula:

$$D_{\vec{u}}f(a,b) = f_{\vec{u}}(a,b) = \nabla f(a,b)\cdot\vec{u} = ||\nabla f(a,b)||||\vec{u}||\cos\theta$$

Properties of the gradient:

1. ∇f is a vector living in the domain of $f(x, y)$.
2. ∇f points in the direction of maximum increase of $f(x, y)$
   Why? $D_{\vec{u}} f$ is maximum if $\cos\theta = 1$ so $\theta = 0$ so $\vec{u} || \nabla f$
3. The direction of maximum decrease (minimum increase) is $-\nabla f$
   because there $\theta = \pi$, $\cos\theta = -1$.
4. The length of ∇f is the maximum rate of increase of f
   because if $\theta = 0$, $D_{\vec{u}} = ||\nabla f||$.
5. ∇f is perpendicular to CONTOURS in the domain because if $\vec{u}$ points to a contour then $D_{\vec{u}} f = 0 = \nabla f \cdot \vec{u}$.

We can draw gradients from contours.

Gradient from contour plot.

Example: $f(x, y) = \sqrt{4 - x^2 - y^2}$ a half-sphere.

$$\nabla f = f_x \vec{i} + f_y \vec{j} = \frac{-x}{\sqrt{4 - x^2 - y^2}} \vec{i} - \frac{y}{\sqrt{4 - x^2 - y^2}} \vec{j} = \frac{1}{\sqrt{4 - x^2 - y^2}} < -x, -y >$$

Gradient of $f(x, y) = \sqrt{4 - x^2 - y^2}$

The gradient is easy to define in higher dimension: $g(x, y, z)$ has gradient $\nabla g = < g_x, g_y, g_z >$
it has all the properties mentioned above.

In particular, if $g(x, y, z) = K$, then $\nabla g \perp$ the level surface $g(x, y, z) = K$.

So we have a vector perpendicular to a surface.
Like a normal to a tangent plane? Yes, exactly!

Say $z = f(x, y)$. Let $g(x, y, z) = z - f(x, y) = 0$

then $\nabla g = < -f_x, -f_y, 1 >$ is the normal to the surface, which is the normal to the tangent plane. So $(a, b, f(a, b))$, the equation of the plane is:

$$-(x-a)f_x(a,b) - (y-b)f_y(a,b) + (z - f(a,b)) = 0$$

Better yet, this works for implicit functions: $2x^2 + y^2 + z^2 = 4 = g(x, y, z)$, an ellipsoid.
We have $\nabla g = < 4x, 2y, 2z >$
The tangent plane at $(1, 1, 1)$ (which is on the surface) is:
Normal is $< 4, 2, 2 >= \hat{n}$.
The plane is $4(x-1) + 2(y-1) + 2(z-1) = 0$ or $4x + 2y + 2z = 8$

so even if $g(x, y, z) = x^{xy^2z^3} + xy - z^2 = 0$, the tangent plane is really easy to find!

Finally, the directional derivative in 3D works the same way

$$g_{\vec{u}}(a, b, c) = D_{\vec{u}}g(a, b, c) = \nabla g(a, b, c) \cdot \vec{u}$$

with $\vec{u}$ a unit vector.