Quiz 4 taken Friday or tomorrow
Quiz 5 taken next

Homework 14 posted, Worksheet 5 posted
Lecture 14 notes posted: Chain rule

Lecture 14, Section 4.5: Chain Rule for grown-ups

Very often, functions depend on variables which themselves depend on other variables.
For example: The price of burritos depends on the price of rice (r) and of salsa (s).

$$B(r,s) = 20r + s^2$$

Off. Hrs: François, today 1-2pm, ACS 362C

Both the prices of rice and salsa depend on time: $r(t) = 1 + t/2$ and $s(t) = 2 + 3t^3$.
So $B(r(t), s(t))$ is really a function of time (only).
Given t, we can find $r(t)$, $s(t)$ and then B:

Cheat Sheet 1 posted
Old exam posted

$$\begin{aligned} B(t) &= = B(r(t), s(t)) = 20r(t) + (s(t))^2 \\ &= 20(1 + t/2) + (2 + 3t^3)^2 \\ &= 20 + 10t + 4 + 12t^3 + 9t^6 \end{aligned}$$

Posted later today:
List of sections covered
Seating assignments

So what is $\frac{dB}{dt}$?
We can calculate it easily from the last formula obtained:

$$\frac{dB}{dt} = 10 + 36t^2 + 54t^5$$

or from the previous one:

$$\frac{dB}{dt} = 20(1/2) + 2(2 + 3t^3) * 9t^2 = 10 + 36t^2 + 54t^5$$

But we can also get it directly from the first line:

$$\frac{dB}{dt} = \frac{\partial B}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial B}{\partial s}\frac{\partial s}{\partial t} = 20(1/2) + 2(2s) * 9t^2 = 10 + 36t^2 + 54t^5$$

This is what I call the chain rule for grown-ups.

What if r and s depend on more than one variable? Say time t and distance to Mexico d.

$$\begin{aligned} r(t,d) &= (1 + t/2)(1 + d^2 - 4d) \\ s(t,d) &= (2 + 3t^3)e^{dt} \end{aligned}$$

So do we plug in again? Yuck, that sounds like a recipe for mistakes. But the chain rule is still the same:

$$\frac{\partial B}{\partial t} = \frac{\partial B}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial B}{\partial s}\frac{\partial s}{\partial t} = 20(1/2)(1 + d^2 - 4d) + 2(2s)e^{dt}(9t^2 + d)$$

and similarly

$$\frac{\partial B}{\partial d} = \frac{\partial B}{\partial r}\frac{\partial r}{\partial d} + \frac{\partial B}{\partial s}\frac{\partial s}{\partial d} = 20(1 + t/2)(2d - 4) + 2(2s)e^{dt}(t)$$

So how do we get these formulas? With a diagram

We need to find all the ways (paths) in which B depends on t and add them up. Here $\frac{dB_1}{dt} = (1) + (2)$.
As you go down, you multiply derivatives:

$$\frac{\partial B}{\partial t} = (1) + (2) + (3) = \frac{\partial B}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial B}{\partial s}\frac{\partial s}{\partial t} + \frac{\partial B}{\partial u}\frac{\partial u}{\partial t}$$

Chain rule diagram

Try $f(x, y, z)$ with $x = r\cos\theta$, $y = r\sin\theta$ and $z = 1 - r^2$. Note that extra connections in your diagram are harmless (they will just give a 0 derivative).

$$\frac{\partial f}{\partial r} = f_x x_r + f_y y_r + f_z z_r = f_x(\cos\theta) + f_y(\sin\theta) + f_z(-2r)$$

$$\frac{\partial f}{\partial \theta} = f_x x_\theta + f_y y_\theta + f_z z_\theta = f_x(-r\sin\theta) + f_y(r\cos\theta) + f_z(0)$$

Does that work with implicit formulas? Yes. Say $f(x, y, z) = g(x, y, z)$ with $x(r, \theta)$ and $y(r, \theta)$.
So implicitly, we must have $z(r, \theta)$. What is $\frac{\partial z}{\partial r}$? We have

$$f_x x_r + f_y y_r + f_z z_r = g_x x_r + g_y y_r + g_z z_r$$

So

$$\frac{\partial z}{\partial r} = \frac{g_x x_r + g_y y_r - f_x x_r + f_y y_r}{f_z - g_z}$$

In general, if we have $F(x, y, z) = 0$, then this formula becomes

$$\frac{\partial z}{\partial r} = -\frac{F_x x_r + F_y y_r}{F_z}$$