Lecture 13, Section 4.4: Tangent planes & Linear Approximations

We know how to calculate the slope in x (labeled $\frac{\partial f}{\partial x}$) and in y (labeled $\frac{\partial f}{\partial y}$). We want to use this to approximate the surface.

Recall the tangent line approximation in 2D
Starting with $f(x)$, we find a linear approximation near the point $(x_0, y_0 = f(x_0))$. Our linear approximation is $l(x) = y_0 + f'(x_0)(x - x_0)$.

Quiz 4 taken today or Tuesday
Quiz 5 posted later today

Cheat Sheet 1 posted
Old exam

Homework 13 posted, Worksheet 5 posted
Lecture 13 notes posted: Tangent planes

Off. Hrs: Lucas today 1-3pm, ACS 362B

Tangent line to a curve, and illustration of a tangent plane to a surface

We would like to use a TANGENT PLANE, $\Pi(x, y)$ to approximate $z = f(x, y)$. We require a few things of this plane:
1) Plane has to agree with the surface at a point $(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$.
So we will have $\Pi(x, y) - z_0 = m(x - x_0) + n(y - y_0)$ or $z = m(x - x_0) + n(y - y_0) + f(x_0, y_0)$.
2) The x-slope of the plane and of the surface should be the same at (x_0, y_0).

$$\frac{\partial \Pi}{\partial x} = m = \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = f_x(x_0, y_0)$$

3) The y-slope of the plane and of the surface should be the same at (x_0, y_0) too.

$$\frac{\partial \Pi}{\partial y} = n = \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = f_y(x_0, y_0)$$

So we have $\Pi(x, y) = z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z_0$.

For example: find the tangent plane to $z = f(x, y) = x^2 + y^2/4$ above (1/2,1)
Plane is $z = (x - 1/2) + 1/2(y - 1) + 1/2$.

A function is said to be differentiable if a tangent plane can be found and if it gives a "good" approximation to $f(x, y)$. Then $\Pi(x, y)$ is a LINEARIZATION of $f(x, y)$.

Example: We had $z = 1/2 + (x - 1/2) + 1/2(y - 1)$ as an approximation to $f(x, y) = x^2 + y^2/4$ near $(1/2, 1)$.

What is $f(0.47, 1.02)$? I don't know exactly, it is too hard.
Our approximation will be: $f(0.47, 1.02) \approx \Pi(0.47, 1.02) = 1/2 - 0.03 + 0.01 = 0.48$.

More precisely, a function is differentiable if the following condition holds

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - \Pi(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

Some example of surfaces that are not differentiable include the tip of a cone and a step.

Theorem: If $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and are continuous, then $f(x, y)$ is differentiable at (x_0, y_0).

We can express our linearisation in terms of differentials (amount of change).
Define $\Delta z = z - z_0$ and $\Delta x = x - x_0$ and $\Delta y = y - y_0$.
Our tangent plane formula is then

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

This formula is exact on the plane, but only approximative for the original function.

If we take the limit of $\Delta x, \Delta y, \Delta z \to 0$, we get the total differential

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

This is an easy way to calculate how much a function changes. In fact it is the same method as using the tangent plane, but we refer to both methods indiscriminately.

Why does it matter? Why bother?
Because real functions, real life, is too hard to handle exactly. So we use linear functions, which we can work with, instead of realistic ones we can't work with.