Lecture 12, Section 4.3: Derivatives of functions of several variables

Recall the definition of a derivative:

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \text{ slope of the tangent}$$

Quiz 4, taken Friday or Tuesday

Lecture notes 12
Homework 12 posted

Practice Exam posted

Office hours: Hannah, today, 1:30-3:30pm ACS-312

Today: Partial Derivatives

Derivative of a function of one variable.

But surfaces don't have one slope. Rather, their slope depends on which way you look at them. Think of a mountain, where you can stay level, or go up, or down.

The slope of a surface depends on the direction in which you look.

Strategy: Consider only one direction at a time, then compute the slope.
How can we easily do that? Recall our traces, where one variable was **fixed**.

So to look in the x-direction, we fix $y = b$.
Then $f(x, y)$ becomes $f(x, b)$, a function of only one variable. We can also denote it $f_b(x)$.

Slope of the tangent in the x-direction

We can now take its derivative. We differentiate only part of $f(x, y)$, so we call it the PARTIAL DERIVATIVE of $f(x, y)$ with respect to x:

$$\left.\frac{\partial f}{\partial x}\right|_{y=b} = f_x(x, b) = f_b'(x) = \lim_{h\to 0} \frac{f(x+h, b) - f(x, b)}{h}$$

Note that the other variable (here y) has to be fixed. This is the slope of the tangent of the trace when $y = b$, or the slope of the surface when looking in the x-direction.

Similarly

$$\left.\frac{\partial f}{\partial y}\right|_{x=a} = f_y(a, y) = f_a'(y) = \lim_{h\to 0} \frac{f(a, y+h) - f(a, y)}{h}$$

You may think of these 2 traces (one with fixed x and one with fixed y) as intersecting roads, with $f(x, y)$ representing the height of the ground.

Standing in front of the library, we'll call x the gym-bound direction and y the library-bound direction. Then $f_y \approx 0$, as this is pretty flat, but $f_x < 0$ as it is going down toward the gym.

A contour diagram allows to ESTIMATE partial derivatives

Using contours to estimate partial derivatives

$$f_x(0, 1) = \frac{f(3, 1) - f(0, 1)}{3} = \frac{2 - 1}{3} = 2/3$$

$$f_y(0, 1) = \frac{f(0, 1.8) - f(0, 1)}{0.8} = \frac{2 - 1}{0.8} = 1.25$$

In practice, we find $f_x(x, y)$ and $f_y(x, y)$ by treating the other variable as a constant. For example, consider $f(x, y) = 3x^2y^4$.
Then $\frac{\partial f}{\partial x} = f_x(3y^4)(2x) = 6xy^4$
and $\frac{\partial f}{\partial y} = f_y(3x^2)(4y^3) = 12x^2y^3$.

Consider $z = f(x, y) = \sqrt{4 - x^2 - y^2}$, the top half of a sphere, and look at the point $(0, 1)$.

$$f_x(1,0) = f_x(x,y)\|_{x=1,y=0} = \left.\frac{-2x}{2\sqrt{4-x^2-y^2}}\right|_{(1,0)} = \frac{-1}{\sqrt{3}} \text{ (down)}$$

$$f_y(1,0) = f_y(x,y)\|_{x=1,y=0} = \left.\frac{-2y}{2\sqrt{4-x^2-y^2}}\right|_{(1,0)} = 0 \text{ (flat)}$$

Slopes on a half-sphere

Could we get the equation of a tangent plane out of those 2 slopes?