Lecture 11, Section 4.1: Multivariable functions

We now study more general two-variable scalar functions. They take 2 inputs and produce a single output: $\mathbb{R}^2 \to \mathbb{R}$ which is written as $(x, y) \to f(x, y)$.

The DOMAIN is the set of all acceptable inputs, living in the xy-plane.
The RANGE or the IMAGE is the set of all possible outputs, living in $\mathbb{R}$ (along the z-axis).

Example: $f(x, y) = \sqrt{x} + \sqrt{3-y}$
So $f(1, 2) = 2$, for example. The domain is $\{x \geq 0, y \leq 3\}$
The range is $\{z \geq 0\}$.

Quiz 3 taken today or Tuesday
Quiz 4 posted later today

Homework 11 posted, Worksheet 4 posted
Lecture 11 notes posted

Off. Hrs: Lucas today 1-3pm, ACS 362B

Domain and range of $f(x, y) = \sqrt{x} + \sqrt{3-y}$

Very often quantities depend on more than 2 variables. For example, your final grade depends on your quizzes and exams. Or the temperature depends on the location in 3D space and on time.

Consider $f(x, y) = \sqrt{y-x}\log(x+y)$.
The domain is given by $y \geq x$ and $x + y > 0$.
The range is $\mathbb{R}$.

Domain and range of $f(x, y) = \sqrt{y-x}\log(x+y)$

Consider $f(x, y) = e^{\sqrt{4-x^2-y^2}}$.
Domain is $\{(x, y) | x^2 + y^2 \leq 4\}$
Range is $\{z | e^0 \leq z \leq e^2\}$.

Let's graph it. Using traces, we find:
if $x = 0$, $z = e^{\sqrt{4-y^2}}$.
if $x = \pm\sqrt{3}$, $z = e^{\sqrt{1-y^2}}$. Note that these traces are 2D curves, like the ones we know well, and we could differentiate them , find their tangent line, etc.
You get similar traces if you fix y, because we have x symmetric to y here.

Domain and range of $f(x, y) = e^{\sqrt{4-x^2-y^2}}$

Looking at level curves $z = 0$ gives nothing.
$z = 1$ gives $\sqrt{4 - x^2 - y^2} = 0$ so $x^2 + y^2 = 4$.
$z = 2$ gives $\sqrt{4 - x^2 - y^2} = \log 2$ so $x^2 + y^2 = 4 - (\log 2)^2$.
so we get smaller and smaller circles.

Traces and level curves of $f(x, y) = e^{\sqrt{4-x^2-y^2}}$

Putting is all together, we get a kind of ”cap” hanging in space.

Surface of $f(x, y) = e^{\sqrt{4-x^2-y^2}}$

Note: What are the contours of $f(x, y) = z = -x - 2y + 4$?
If we fix $z = k$, we get $k = -x - 2y + 4$, which are straight lines (figure).
A good way to visualize a plane is to find where it intersects the x, y, and z planes and connect the triangle.
Here we have (figure)
$x = y = 0$ gives $z = 4$
$x = z = 0$ gives $y = 2$
$y = z = 0$ gives $x = 4$

Level curves and surface of $f(x, y) = z = -x - 2y + 4$

We can also try to represent 3-variable functions $w = g(x, y, z)$. The domain is then a portion of the 3D space, and the whole ”hyper-surface” requires 4 dimensions to be represented. This is hard (but you could make a movie!)

Example: $w = f(x, y, z) = (x - 1)^2 + y^2 + z^2$. If we look at level surfaces, we would get:
$w = k = (x - 1)^2 + y^2 + z^2$.
so if $k = 0$, we get a point. If k increases, we get spheres of radius $\sqrt{k}$. This can be represented with a

Level surfaces of $w = f(x, y, z) = (x - 1)^2 + y^2 + z^2$

movie. More importantly, we can still do calculus on such an object.