Lecture 10, Section 3.1, 3.2, 3.3, 3.4: Examples of Parametrizations

Let us now look at a few examples of vector valued functions. We will look for parametrizations of curves. A parametrization is a specific equation, or equations describing a curve in terms of a parameter.

Quiz 3, taken Fri. or Tue.

Cheat Sheet 1 posted

Lecture notes 10 today: Examples of parametrizations
Homework 10 posted

No class on Monday

Off. hrs: Today, Hannah, 1:30-2:30pm, 3:45-4:45pm, ACS 312
Friday, Francois, 9-10am, ACS 362C

The cycloid

Example 1: Cycloid.
Consider a point fixed on a wheel of radius R rolling on a horizontal surface. What is the position of the point over time? (figure)
Assume the angle to the horizontal is changing at a constant rate $\theta = \omega t$. We then have the position $\vec{r}(t) =< x(t), y(t) >$

$$\begin{aligned} x(t) &= -R\theta + R\cos\theta = -R\omega t + R\cos\omega t \\ y(t) &= R\sin\theta = R\sin\omega t \end{aligned}$$

The velocity is then $\vec{r}\,'(t) =< x'(t), y'(t) >$

$$\begin{aligned} x'(t) &= -R\omega - \omega R\sin\omega t \\ y;(t) &= \omega R\cos\omega t \end{aligned}$$

with a speed of $|\vec{r}\,'(t)| = R\omega\sqrt{2 + 2\sin\omega t} = R\omega\sqrt{2}|\cos\omega t/2 + \sin\omega t/2|$.

We can calculate the arclength, if $\cos\omega t/2 + \sin\omega t/2 > 0$ or if $-\pi/4 < \omega t/2 < 3\pi/4$ or $-\pi/2 < \omega t < 3\pi/2$.
Then we have

$$s = \int_0^t d\tau|\vec{r}\,'\tau| = \int_0^t d\tau R\omega\sqrt{2}\cos\omega\tau/2 + \sin\omega\tau/2 = \sqrt{2}R\omega(2/\omega)(-\cos\omega t/2 + \sin\omega t/2)|_0^t = 2\sqrt{2}R(1 - \cos\omega t/2 + \sin\omega t/2)$$

The acceleration is $\vec{r}\,''(t) = R\omega^2 < -\cos\omega t, -\sin\omega t >$. This always points toward the center of the circle.

Note that $\vec{N} = |\frac{d\vec{T}}{dt}|^{-1}\frac{d\vec{T}}{dt}$ is the unit normal and
the acceleration has a normal component and a tangential component"

$$\vec{r}\,'' = r_N\vec{N} + r_T\vec{T}.$$

Second example: Projectile, a classic. Say that North is the y-axis.
A missile is fired toward the NW, with an angle of $\pi/3$ from the vertical with an initial velocity of 240m/s.
1) Calculate its trajectory if air has been sucked out by the army (temporarily)
2) Determine where it will land.

A projectile in 2D and 3D

We want $\vec{r}(t)$. What do we have?
$\vec{r}\,''(t) =< 0, 0, -g >\approx< 0, 0, -10 >$m/s^2
$\vec{r}\,'(0) = 240$m/s$< \sin\pi/3\cos 3\pi/4, \sin\pi/3\sin 3\pi/4, \cos\pi/3 >$
$\vec{r}\,'(0) = 240$m/s$< -\sqrt{6}/4, \sqrt{6}/4, 1/2 >$
We will assume for convenience that $\vec{r}(0) =< 0, 0, 0 >$.

Integrating the acceleration, we find
$\vec{r}\,'(t) = \vec{r}\,'(0)+ < 0, 0, -10t >=< -60\sqrt{6}, 60\sqrt{6}, 120 - 10t >$m/s
And integrating the velocity we found, we get
$\vec{r}(t) = \vec{r}(0)+ < -60\sqrt{6}t, 60\sqrt{6}t, 120t - 5t^2 >=< -60\sqrt{6}t, 60\sqrt{6}t, 120t - 5t^2 >$.
So we found an equation for the trajectory.

Where does it land? Wherever $z(t) = 0$. So 120t = $5t^2$ and $t = 24$s or $t = 0$.
The position is then $< -60\sqrt{6}\,24, 60\sqrt{6}\,24, 0 >$.
How far is that? $24 * 60 * \sqrt{12} \approx 5$km.

What is the speed? $v = (60^2 6 + 60^2 6 + (120 - 10t)^2)^{1/2} = 10(576 - 24t + t^2)^{1/2}$

The arclength? $s = \int_0^t 10(576 - 24\tau + \tau^2)^{1/2} d\tau$
Highest point? It is where $z'(t) = 0$ so when $120 = 10t$ so when $t = 12$.
So the point is $\vec{r}(12) =< -720\sqrt{6}, 720\sqrt{6}, 720 >$m. So the maximum height is 720m.

The unit tangent vector is

$$\hat{T} = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{< -60\sqrt{6}, 60\sqrt{6}, 120 - 10t >}{10(576 - 24t + t^2)^{1/2}} = \frac{< -6\sqrt{6}, 6\sqrt{6}, 12 - t >}{(576 - 24t + t^2)^{1/2}}$$

The normal direction is

$$\vec{N} = \frac{d\hat{T}}{dt} =< \frac{6\sqrt{6}(t-12)}{(576 - 24t + t^2)^{3/2}}, \frac{-6\sqrt{6}(t-12)}{(576 - 24t + t^2)^{3/2}}, \frac{-(12-t)(t-12)}{(576 - 24t + t^2)^{3/2}} - \frac{1}{(576 - 24t + t^2)^{1/2}} >$$

and we could also find the unit normal as $\hat{N} = \vec{N}/\|\vec{N}\|$. Once again, we have

$$\vec{r}\,'' = r_N\hat{N} + r_T\hat{T}.$$

with part of the acceleration changing the speed of the projectile, and part changing its direction.