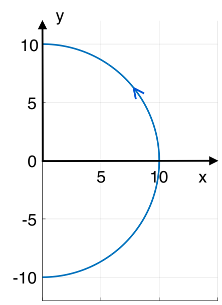
Duration: 50 minutes
Instructions: Answer all questions, without the use of notes, books or calculators. Partial credit will be awarded for correct work, unless otherwise specified. The total number of points is 100.

1. (24 pts: 12, 12) Give an integral that computes the mass of a solid bounded below by the plane $z = 0$ and above by $z = \sqrt{9 - x^2 - y^2}$ and whose density is $d(x, y, z) = (z + 1)(x^2 + y^2 + z^2)$. DO NOT EVALUATE.

   a) in Cylindrical coordinates b) in Spherical coordinates.

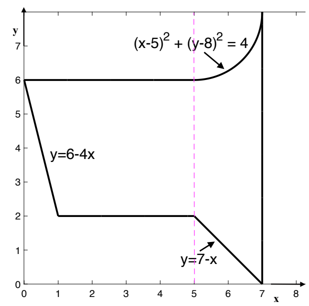
2. (24 pts: 8, 8, 8) The wind velocity field over Merced is given by $\vec{F} =< 2x \sin y + 2, x^2 \cos y >$. A drone is flying along a half circle of radius 10 (units are kilometers), counterclockwise, with constant height $z = 0.1$, as shown in the figure below.



   (a) Determine if the wind velocity field $\vec{F}$ is conservative.
   (b) If $\vec{F}$ is conservative, find its potential ϕ. If $\vec{F}$ is not conservative, parametrize C.
   (c) Give an applicable method to compute the work done by $\vec{F}$ on the drone (do not have to perform the computation).

3. (24 pts: 8, 16) The integral $\int\int_D f(x, y)\, \mathrm{d}A$ calculates the total amount of pollution above the United States on April 1, 2026.

   (a) Give the appropriate meaning of D, and of the function $f(x, y)$.
   (b) If D is approximated as the region drawn below, set-up the proper bounds of integration in Cartesian coordinates by integrating y first.



4. (28pts, 7 each) Answer the following in no more than two lines of text or computations.

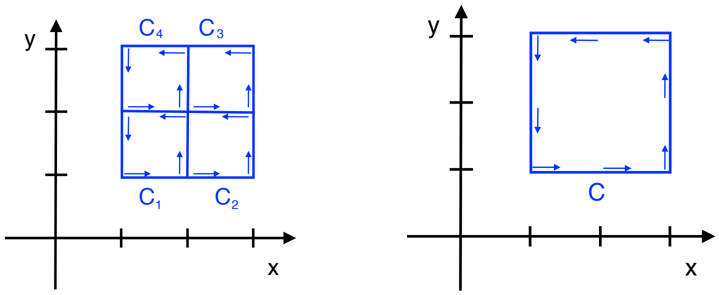
 (a) The integral $\int_C h(x,y)\, \mathrm{d}s$ represents the total gas consumption of a car traveling from Los Angeles to Seattle. Give relevant meanings to C and $h(x,y)$.

 (b) The work done over a contour C by a vector field $\vec{F}$ was computed using Green's theorem and the result was 2. Sketch in the xy-plane an example of what C might be.

 (c) A function $f(x,y)$ has the following properties at the point (a,b):
 (i) $f_x(a,b) = 1$ (ii) $f_y(a,b) = 2$ (iii) $f_{xx}(a,b) = 3$ (iv) $f_{yy}(a,b) = 4$ (v) $f_{xy}(a,b) = 5$.

 What can you conclude with regards to the optimality of $f(a,b)$?
 (A) The point (a,b) is a local maximum of $f(a,b)$
 (B) The point (a,b) is a local minimum of $f(a,b)$
 (C) The point (a,b) is a saddle-point of $f(a,b)$
 (D) The point (a,b) is not a local maximum, local minimum, or saddle-point of $f(a,b)$
 (E) There is not enough information to conclude anything (the test was inconclusive).

 (d) Using the contours shown in the figure below and a continuously differentiable $\vec{F}$, define $\alpha = \int_C \vec{F} \cdot d\vec{r}$ and $\beta = \int_{C_1+C_2+C_3+C_4} \vec{F} \cdot d\vec{r}$. Which statement is true?



 (A) $\alpha - \beta > 0$
 (B) $\alpha - \beta = 0$
 (C) $\alpha - \beta < 0$
 (D) The sign of $\alpha - \beta$ depends on $\vec{F}$
 (E) I don't think this is the correct answer but I am aura farming

Dot product: $\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z = ||\vec{u}||\, ||\vec{v}|| \cos\theta_{uv}$

Cross product: $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$ and $||\vec{u} \times \vec{v}|| = ||\vec{u}||\, ||\vec{v}||| \sin\theta_{uv}|$

Line through P in direction $\vec{v}$: $\vec{x}(t) = P + \vec{v}\, t$

Plane through P with normal $\vec{n}$: $\vec{n} \cdot (\vec{x} - P) = 0$

Projection of $\vec{v}$ onto $\vec{w}$: $\text{Proj}_{\vec{w}}\, \vec{v} = \frac{\vec{v} \cdot \vec{w}}{||\vec{w}||} \frac{\vec{w}}{||\vec{w}||}$

General Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

General Hyperboloid: $\pm \frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = \pm 1$ not all signs the same

General Paraboloid: $z = \pm(\frac{x^2}{a^2} + \frac{y^2}{b^2}) + h$

General Hyperbolic paraboloid: $z = \pm(\frac{x^2}{a^2} - \frac{y^2}{b^2}) + h$

Position: $\vec{x}(t)$, Velocity: $\frac{d\vec{x}}{dt}$, Acceleration: $\frac{d^2\vec{x}}{dt^2}$, speed: $\left|\left|\frac{d\vec{x}}{dt}\right|\right|$

Arclength: $s = \int_C ds = \int_{t_i}^{t_f} \left|\left|\frac{d\vec{x}}{dt}\right|\right| dt$

When plotting $z = f(x, y)$:

$C = f(x, y)$ is a contour line in the domain,

$z = f(x, C)$ is a vertical trace in space

Partial derivative of $f(x, y)$: $\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$

Plane tangent to $z = f(x, y)$ at (x_0, y_0, z_0): $\Pi(x, y) = z_0 + (x - x_0) \left(\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} \right) + (y - y_0) \left(\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} \right)$

Chain Rule for $f(x, y)$ with $x(s, t, u)$, and $y(s, t, u)$: $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$

Gradient of $f(x, y)$: $\text{grad} f = \nabla f = < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$

Min/Max of $f(x, y)$: $P = (x_0, y_0)$ is a critical point if $\frac{\partial f}{\partial x}(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) = 0$

If, in addition, $D = f_{xx}(P) f_{yy}(P) - (f_{xy}(P))^2 < 0$, then P is a saddle-point

if $D > 0$ and $f_{xx}(P) > 0$, then P is a local minimum

if $D > 0$ and $f_{xx}(P) < 0$, then P is a local maximum

For a double integrals: $\int\int f(x, y) \mathrm{d}A$ the area element $\mathrm{d}A$ is

$\mathrm{d}A = \mathrm{d}x\mathrm{d}y$ in Cartesian coordinates

$\mathrm{d}A = r\, \mathrm{d}r\mathrm{d}\theta$ in polar coordinates

For a triple integrals: $\int\int\int f(x, y, z) \mathrm{d}V$ the volume element $\mathrm{d}V$ is

$\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ in Cartesian coordinates

$\mathrm{d}V = r\, \mathrm{d}r\mathrm{d}\theta\mathrm{d}z$ in cylindrical coordinates

$\mathrm{d}V = \rho^2 \sin\phi\, \mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta$ in spherical coordinates.

Coordinate conversion:

For polar and cylindrical: $x = r\cos\theta$, $y = r\sin\theta$

For spherical: $r = \rho\sin\phi$, $z = \rho\cos\phi$

Line Integral $\int_C f(\vec{x}) \mathrm{d}s = \int_{t_i}^{t_f} f(\vec{x}(t)) \left|\left|\frac{d\vec{x}}{dt}\right|\right| \mathrm{d}t$

Fundamental Theorem. of Line Integrals: if $\vec{F} = \nabla\phi$ and C goes from A to B, then

$\int_C \vec{F}(\vec{x}) \cdot \frac{d\vec{x}}{dt} \mathrm{d}t = \phi(B) - \phi(A)$

Green's Theorem: $\int_C \vec{F}(\vec{x}) \cdot \frac{d\vec{x}}{dt} \mathrm{d}t = \int\int_D \left(\frac{dF_2}{dx} - \frac{dF_1}{dy} \right) dA$