Physics 9 Fall 2011  
Homework 2 - Solutions  
Friday September 2, 2011

Make sure your name is on your homework, and please box your final answer. Because we will be giving partial credit, be sure to attempt all the problems, even if you don’t finish them. The homework is due at the beginning of class on Friday, September 9th. Because the solutions will be posted immediately after class, no late homeworks can be accepted! You are welcome to ask questions during the discussion session or during office hours.

1. Consider an infinite number of identical charges, each of charge \( q \), placed along the \( x \) axis at distances \( a, 2a, 3a, 4a, \ldots \), from the origin. What is the electric field at the origin due to this distribution? 

   \[ 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}. \]

**Solution**

The net electric field is just the sum of the fields from each charge. So, since each field is given by the Coulomb law, then the net charge is

\[
E_{\text{net}} = \frac{kq}{a^2} + \frac{kq}{(2a)^2} + \frac{kq}{(3a)^2} + \frac{kq}{(4a)^2} + \cdots ,
\]

which we can write as

\[
E_{\text{net}} = \frac{kq}{a^2} \left[ 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \right].
\]

According to the hint we can write the term in brackets as \( \pi^2/6 \), and so the total field is

\[
E_{\text{net}} = \frac{\pi^2 kq}{6a^2}.
\]
2. A charge of 3.00 µC is uniformly distributed on a ring of radius 10 cm. Find the electric field strength on the axis at distances of (a) 1.0 cm, (b) 10.0 cm, and (c) 5.0 m from the center of the ring. (d) Find the field strength 5.0 m using the approximation that the ring is a point charge at the origin, and compare your results for parts (c) and (d). Is your approximation a good one? Explain your answer.

Solution

The electric field of a charged ring of charge $Q$ and radius $R$ is

$$E(z) = \frac{1}{4\pi\varepsilon_0} \frac{zQ}{(z^2 + R^2)^{3/2}}.$$

We just need to plug in the distances along $z$.

(a) Taking $z = 0.01$ m, then

$$E(z) = \frac{1}{4\pi\varepsilon_0} \frac{zQ}{(z^2 + R^2)^{3/2}} = \frac{9 \times 10^9 \times 0.01 \times 3 \times 10^{-6}}{(0.01^2 + 0.1^2)^{3/2}} = 2.66 \times 10^5 \text{ N/C}.$$

(b) Next, with $z = 0.1$ m, we have

$$E(z) = \frac{1}{4\pi\varepsilon_0} \frac{zQ}{(z^2 + R^2)^{3/2}} = \frac{9 \times 10^9 \times 0.1 \times 3 \times 10^{-6}}{(0.1^2 + 0.1^2)^{3/2}} = 9.55 \times 10^5 \text{ N/C}.$$

(c) Finally, with $z = 5$ m, we find

$$E(z) = \frac{1}{4\pi\varepsilon_0} \frac{zQ}{(z^2 + R^2)^{3/2}} = \frac{9 \times 10^9 \times 5 \times 3 \times 10^{-6}}{(5^2 + 0.1^2)^{3/2}} = 1079 \text{ N/C}.$$

(d) Treating the ring like a point charge we just have Coulomb’s law,

$$E(z) = \frac{1}{4\pi\varepsilon_0} \frac{q}{z^2} = \frac{9 \times 10^9 \times 3 \times 10^{-6}}{5^2} = 1080 \text{ N/C},$$

which is extremely close to the actual answer found in part (c). Thus, this approximation is excellent at these large distances. This makes sense since the details of the charge distribution are not seen at large distances, and all we see is basically a point charge.
3. Show that the electric field strength $E$ on the axis of a ring charge of radius $R$ has maximum values at $z = \pm R/\sqrt{2}$. What is this maximum value of $E$?

Solution

The electric field along the axis of a ring of charge $Q$ is given by $E(z) = \frac{1}{4\pi\varepsilon_0} \frac{zQ}{(z^2 + R^2)^{3/2}}$, where $z$ is the distance along the axis, and $R$ is the radius of the ring. So, the field depends on $z$. It has a maximum when $\frac{dE}{dz} = 0$. Taking the derivative gives

$$\frac{dE}{dz} = \frac{Q}{4\pi\varepsilon_0} \frac{dz}{dz} \left( \frac{z}{(z^2 + R^2)^{3/2}} \right) = \frac{Q}{4\pi\varepsilon_0} \left( \frac{1}{(z^2 + R^2)^{3/2}} - \frac{3}{2} \frac{2z^2}{(z^2 + R^2)^{5/2}} \right) = 0$$

This is zero when the term inside the parenthesis vanishes. This gives $z^2 + R^2 = 3z^2$, or $z = \pm \frac{R}{\sqrt{2}}$, as claimed.

When $z = \pm \frac{R}{\sqrt{2}}$, then

$$E\left(\pm \frac{R}{\sqrt{2}}\right) = \frac{1}{4\pi\varepsilon_0} \frac{(\pm R/\sqrt{2}) Q}{\left(\pm R/\sqrt{2}\right)^2 + R^2} = \pm \frac{1}{4\pi\varepsilon_0} \frac{RQ}{\sqrt{2} (3R^2/2)^{3/2}} = \pm \frac{2}{3\sqrt{3}} \frac{1}{4\pi\varepsilon_0} \frac{Q}{R^2}$$

Thus, the maximum value of the electric field is $E = \pm \frac{2}{3\sqrt{3}} \frac{1}{4\pi\varepsilon_0} \frac{Q}{R^2}$. 
4. A charge $+q$ of mass $m$ is free to move along the $x$ axis. It is in equilibrium at the origin, midway between a pair of identical point charges, $+Q$, located on the $x$ axis at $x = +a$ and $x = -a$. The charge at the origin is displaced a small distance $x \ll a$ and released. Show that it can undergo simple harmonic motion with an angular frequency

$$
\omega_0 = \sqrt{\frac{4kqQ}{ma^3}}.
$$

Hint: use the binomial expansion.

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Solution

When the charge $q$ is at the origin, it is in stable equilibrium since the net force on it is zero. The charge $Q$ at $x = -a$ pushes $q$ to the right, while the charge $Q$ at $x = +a$ pushes it back to the left, both with equal magnitude. So, at equilibrium,

$$
F_{\text{net}} = \frac{kqQ}{a^2} - \frac{kqQ}{a^2} = 0.
$$

Now, suppose that we shift the charge $q$ to the right by a small amount $x$. Then the force from the left is a little bit smaller, while the force from the right is a little bit bigger. Hence, the net force is

$$
F_{\text{net}} = \frac{kqQ}{(a + x)^2} - \frac{kqQ}{(a - x)^2},
$$

which can be rewritten as

$$
F_{\text{net}} = \frac{kqQ}{a^2} \left[ \left(1 + \frac{x}{a}\right)^{-2} - \left(1 - \frac{x}{a}\right)^{-2} \right].
$$

Now, since $x \ll a$, we can expand this answer as

$$
F_{\text{net}} \approx \frac{kqQ}{a^2} \left[ 1 - \frac{2x}{a} - 1 - \frac{2x}{a} \right] = -\frac{4kqQ}{a^3} x.
$$

But, the force is $F = ma = m \frac{d^2x}{dt^2}$, so the acceleration is

$$
\frac{d^2x}{dt^2} = -\frac{4kqQ}{ma^3} x,
$$

which is the equation for a simple harmonic oscillator with angular frequency

$$
\omega_0 = \sqrt{\frac{4kqQ}{ma^3}}.
$$
5. The field of an electric dipole decreases as \(1/r^3\) when the distance of a given point to the dipole, \(r\), is much larger than the separation between the charges. The only way to arrange two charges with a total charge of zero is to form a dipole. There are, however, many ways to arrange four charges with a total charge of zero in a compact pattern. An arrangement with an electric field that behaves at great distances as \(1/r^4\) is an electric quadrupole. For four charges aligned with alternating signs (such as \(+−−+\), so that the combination acts like dipoles of opposite orientation) along an axis, show that the field on the axis perpendicular to the line of charges decreases as \(1/r^4\), where \(r\) is much larger than any separation distance within the quadrupole. 

**Solution**

Consider the quadrupole seen in the figure to the right, where the dark spheres are \(+q\), and the light ones are \(−q\). The charges are all separated by a distance \(a\), and we are interested in the field at a point \(r ≫ a\). The net field of the distribution is the sum of the fields of each charge. It’s easy to see from the symmetry of the situation that the net field points entirely along the vertical direction, towards the quadrupole. Hence we only need the vertical components. The vertical fields from the two negative charges are

\[
E_− = −\frac{2kq}{(r^2 + (a/2)^2)} \cos θ_1 = −\frac{2kqr}{(r^2 + (a/2)^2)^{3/2}},
\]

where we have recalled that \(\cos θ_1 = r/\sqrt{r^2 + (a/2)^2}\). Now, the vertical field from the two positive charges is found in the same way

\[
E_+ = \frac{2kq}{(r^2 + (3a/2)^2)} \cos θ_2 = \frac{2kqr}{(r^2 + (3a/2)^2)^{3/2}},
\]

So, the total field is just the sum of these two fields, which we can find using the binomial expansion,

\[
E = −2kqr \left[ \frac{1}{(r^2 + (a/2)^2)^{3/2}} - \frac{1}{(r^2 + (3a/2)^2)^{3/2}} \right]
\]

\[
= −\frac{2kq}{r^2} \left[ \frac{1}{(1+(a/2r)^2)^{3/2}} - \frac{1}{(1+(3a/2r)^2)^{3/2}} \right]
\]

\[
= −\frac{2kq}{r^2} \left[ \left(1 + \frac{a^2}{2r^2}\right)^{-3/2} - 1 + \left(\frac{3a^2}{2r^2}\right)^{-3/2} \right]
\]

\[
≈ −\frac{2kq}{r^2} \left[ 1 - \frac{3}{2} \left(\frac{a}{2r}\right)^2 - 1 + \frac{3}{2} \left(\frac{3a}{2r}\right)^2 \right]
\]

\[
= −\frac{3kq}{r^2} \left[ 9a^2/4r^2 - a^2/4r^2 \right]
\]

\[
= −\frac{6kqa^2}{4r^4},
\]

and does go as \(1/r^4\), as advertised.