Make sure your name is on your homework, and please box your final answer. Because we will be giving partial credit, be sure to attempt all the problems, even if you don’t finish them. The homework is due at the beginning of class on Wednesday, November 23rd. Because the solutions will be posted immediately after class, no late homeworks can be accepted! You are welcome to ask questions during the discussion session or during office hours.

1. Show explicitly that the following functions satisfy the wave equation, 
\[ \frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} = 0. \]

(a) \( y(x, t) = A(x + vt)^3 \),
(b) \( y(x, t) = Ae^{i(kx - \omega t)} \), and
(c) \( y(x, t) = A\ln[kx - \omega t] \), where \( A \) and \( k \) are constants, \( i \equiv \sqrt{-1} \) and \( \omega/k = v \).

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**Solution**

We just need to plug these solutions in to the wave equation and check them.

(a) We just proceed along,
\[
\begin{align*}
\frac{\partial y}{\partial x} &= 3A(x + vt)^2 \\
\frac{\partial y}{\partial t} &= 6A(x + vt) \\
\frac{\partial^2 y}{\partial x^2} &= 6v^2 A(x + vt) \\
\frac{\partial^2 y}{\partial t^2} &= 6v^2 A(x + vt)
\end{align*}
\]

Thus,
\[
\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} = 6A(x + vt) - \frac{1}{v^2} (6v^2 A(x + vt)) = 0.
\]

So, \( y(x, t) = A(x + t)^3 \) satisfies the wave equation.

(b) Next,
\[
\begin{align*}
\frac{\partial y}{\partial x} &= i k A e^{i(kx - \omega t)} \\
\frac{\partial y}{\partial t} &= -k^2 A e^{i(kx - \omega t)} \\
\frac{\partial^2 y}{\partial x^2} &= -i \omega A e^{i(kx - \omega t)} \\
\frac{\partial^2 y}{\partial t^2} &= -\omega^2 A e^{i(kx - \omega t)}
\end{align*}
\]

Thus,
\[
\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} = -k^2 A e^{i(kx - \omega t)} - \frac{1}{v^2} (-\omega^2 A e^{i(kx - \omega t)}) = 0,
\]

where we have recalled that \( v^2 = \omega^2/k^2 \). So, \( y(x, t) = Ae^{i(kx - \omega t)} \) satisfies the wave equation.
(c) Finally,

\[
\begin{align*}
\frac{\partial y}{\partial x} &= \frac{kA}{kx-\omega t} \\
\frac{\partial y}{\partial x^2} &= \frac{-k^2 A}{(kx-\omega t)^2} \\
\frac{\partial y}{\partial t} &= \frac{-\omega A}{kx-\omega t} \\
\frac{\partial^2 y}{\partial t^2} &= \frac{-\omega^2 A}{(kx-\omega t)^2}
\end{align*}
\]

Thus,

\[
\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} = -\frac{k^2 A}{(kx-\omega t)^2} - \frac{1}{v^2} \left( -\frac{\omega^2 A}{(kx-\omega t)^2} \right) = 0,
\]

where we have once again noted that \( v^2 = \omega^2/k^2 \). So, \( y(x, t) = A \ln [kx - \omega t] \) satisfies the wave equation.
2. You have just been pulled over for running a red light, and the police officer has informed you that the fine will be $250. In desperation, you suddenly recall an idea that your physics professor recently discussed in class. In your calmest voice, you tell the officer that the laws of physics prevented you from knowing that the light was red. In fact, as you drove toward it, the light was Doppler shifted to where it appeared green to you. “OK,” says the officer, “Then I’ll ticket you for speeding. The fine is $1 for every 1 km/hr over the posted speed limit of 50 km/hr.” How big is your fine? Use 650 nm as the wavelength of red light and 540 nm as the wavelength of green light.

Solution

Since we are approaching the the light, the wavelengths are shortened due to the Doppler shift. The decrease in the wavelength is given by

$$\lambda = \sqrt{\frac{c-v}{c+v}} \lambda_0 = \sqrt{\frac{1-v/c}{1+v/c}} \lambda_0.$$

Once again, we can solve for the velocity as

$$\frac{v}{c} = \frac{\lambda_0^2 - \lambda^2}{\lambda_0^2 + \lambda^2}.$$

Taking $\lambda_0 = 650$ nm and $\lambda = 540$ nm, we find

$$\frac{v}{c} = \frac{650^2 - 540^2}{650^2 + 540^2} = 0.183.$$

Thus, the velocity is $v = 0.183c \approx 55.5 \times 10^7$ m/s, which is about $2 \times 10^8$ km/hr. This is well above the posted speed limit of 50 km/hr, and so the fine would basically be $200$ million!!! You’re far better off just paying the $250$ fine for running the red light!
3. A string that has a linear mass density of $4.00 \times 10^{-3}$ kg/m is under a tension of 360 N and is fixed at both ends. One of its resonance frequencies is 375 Hz. The next higher resonance frequency is 450 Hz.

(a) What is the fundamental frequency of this string?
(b) Which harmonics have the given frequencies?
(c) What is the length of the string?

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**Solution**

(a) Although we don’t know what mode the 375 Hz oscillation is resonating in, we do know that it must be an integer multiply of the fundamental frequency, $f_1$; call this resonant frequency $nf_1$. The next higher frequency is, then, $(n + 1)f_1$. Now, subtract these two frequencies to get the resonant frequency,

$$(n + 1)f_1 - nf_1 = f_1 = 450 - 375 = 75 \text{ Hz}.$$

Thus, the fundamental frequency is $f_1 = 75 \text{ Hz}$.

(b) Dividing $375/75 = 5$ tells us that the 375 Hz oscillation is resonating in the fifth harmonic, and so the 450 Hz oscillation is resonating in the sixth harmonic.

(c) The fundamental frequency corresponds to a wavelength of twice the length of the string, and since $\lambda f = v$, the speed of the wave which is also $\sqrt{T/\mu}$, we can write

$$\lambda_1 = 2L = \frac{v}{f_1} = \frac{1}{f_1} \sqrt{\frac{T}{\mu}}.$$

Thus, the length of the string is

$$L = \frac{1}{2f_1} \sqrt{\frac{T}{\mu}} = \frac{1}{2 \times 75} \sqrt{\frac{360}{4 \times 10^{-3}}} = 2 \text{ meters}.$$
4. A standing wave on a rope is represented by the wave function

\[ y(x, t) = (0.020) \sin \left( \frac{1}{2} \pi x \right) \cos(40\pi t), \]

where \( x \) and \( y \) are in meters, and \( t \) is in seconds. (a) Write wave functions for two traveling waves that, when superimposed, produce this standing-wave pattern. (b) What is the distance between the nodes of the standing wave? (c) What is the maximum speed of the rope at \( x = 1.0 \text{ m} \)? (d) What is the maximum acceleration of the rope at \( x = 1.0 \text{ m} \)?

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**Solution**

(a) Recall that the sum of two sine waves is

\[ \sin A + \sin B = 2 \cos \left( \frac{1}{2} (A - B) \right) \sin \left( \frac{1}{2} (A + B) \right). \]

This is precisely the correct form of the given standing wave, so we can write

\[ y(x, t) = 0.010 \sin \left( \frac{\pi}{2} x - 40\pi t \right) + 0.010 \sin \left( \frac{\pi}{2} x + 40\pi t \right), \]

which gives the superposition of a wave traveling to the right and another traveling to the left.

(b) The distance between the nodes is just half the wavelength. Now, the wave vector \( k = 2\pi/\lambda = \pi/2 \), so \( \lambda = 4 \text{ meters} \), and thus the distance between the nodes is \( d = 2 \text{ meters} \).

(c) The speed of the rope (not the wave) is just the derivative of the displacement, \( v = \dot{y} \). Taking the derivative gives

\[ v = \frac{d}{dt} \left( (0.020) \sin \left( \frac{1}{2} \pi x \right) \cos(40\pi t) \right) = (0.020) \sin \left( \frac{1}{2} \pi x \right) \frac{d}{dt} [\cos(40\pi t)] = -0.0800\pi \sin \left( \frac{1}{2} \pi x \right) \sin(40\pi t). \]

Now, the maximum speed occurs when the cosine is \(-1\) (to cancel the minus sign in the velocity). So, the maximum velocity is \( v = 0.8\pi \sin \left( \frac{\pi}{2} x \right) \). When \( x = 1 \), then

\[ v(x = 1) = 0.8\pi \sin \left( \frac{\pi}{2} \right) = 0.8\pi \text{ m/s} = 2.5 \text{ m/s}. \]

(d) The acceleration is just the second derivative of the position, \( a = \ddot{y} \), or

\[ a = \frac{d^2}{dx^2} \left( (0.020) \sin \left( \frac{1}{2} \pi x \right) \cos(40\pi t) \right) = (0.020) \sin \left( \frac{1}{2} \pi x \right) \frac{d^2}{dx^2} [\cos(40\pi t)] = -32\pi^2 \sin \left( \frac{1}{2} \pi x \right) \sin(40\pi t). \]

Again, the maximum acceleration occurs when the cosine is \(-1\), and when \( x = 1 \), we find \( a_{\text{max}} = 32\pi^2 \text{ m/s}^2 \).
5. There is a useful quantity in astronomy related to the Doppler shift of light called redshift, \( z \), which is defined as

\[
z + 1 = \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} \tag{1}
\]

where \( \lambda_{\text{obs}} \) is the observed wavelength of the source, and \( \lambda_{\text{emit}} \) is its wavelength at emission.

(a) Galaxies emit electromagnetic radiation from the transition of electrons between different atomic orbits in hydrogen. One such transition is called the Lyman-\( \alpha \) transition. When this transition occurs in hydrogen in the lab it produces light with wavelength \( \lambda_{\text{emit}} = 121.567 \text{ nm} \). This transition has been observed in one of the most distant objects ever seen, called a quasar. The observed wavelength of the Lyman-\( \alpha \) transition from the quasar is \( \lambda_{\text{obs}} = 866.0 \text{ nm} \). What is the redshift of the quasar?

(b) The quasar is moving away from us. Using the formula for the Doppler shift of light, show that the velocity of the source can be written in terms of the redshift as

\[
v = \left[ \frac{(1 + z)^2 - 1}{(1 + z)^2 + 1} \right] c. \tag{2}
\]

(c) What is the speed of recession of the quasar?

(d) It is found that distant galaxies are moving away from us with a speed proportional to their distance from us, \( v = H_0 d \), where \( H_0 \) is called Hubble’s constant. The quasar is at a distance of about 4000 Mpc (1 Mpc, or megaparsec, is about \( 3.1 \times 10^{21} \text{ m} \)). What is Hubble’s constant?

(e) An excellent estimation of the age of the Universe is \( t_0 = \frac{1}{H_0} \). What is this value in years?

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Solution

(a) From the definition of redshift we find \( z + 1 = \lambda_{\text{obs}}/\lambda_{\text{emit}} = 866.0/121.567 = 7.12 \), and so \( z = 6.12 \). This is actually a huge redshift, as this is one of the most distant objects ever found.

(b) Recall that for a source moving away from you at speed \( v \), the Doppler shift for light says

\[
\lambda_{\text{obs}} = \sqrt{\frac{1 + v/c}{1 - v/c}} \lambda_{\text{emit}}. \tag{3}
\]

We just need to play the algebra games to get the answer. First, rewrite the expression by dividing by \( \lambda_{\text{emit}} \), and then substitute in for \( 1 + z \) to get \( 1 + z = \sqrt{\frac{1 + v/c}{1 - v/c}} \). Squaring both sides and rearranging gives \( (1 - v/c)(1 + z)^2 = 1 + v/c \), or \( (1 + z)^2 - v/c(1 + z)^2 = 1 + v/c \). Finally, factoring out the \( v/c \) terms we find
\[
(1 + (1 + z)^2) \frac{v}{c} = (1 + z)^2 - 1, \text{ or } \frac{v}{c} = \frac{(1+z)^2-1}{(1+z)^2+1}. \text{ Multiplying by } c \text{ gives our answer,}
\]

\[
v = \left[ \frac{(1 + z)^2 - 1}{(1 + z)^2 + 1} \right] c.
\]

(c) We just need to plug in for the redshift.

\[
v = \left[ \frac{(1 + z)^2 - 1}{(1 + z)^2 + 1} \right] c = \frac{7.12^2 - 1}{7.12^2 + 1} c = 0.96c = 2.88 \times 10^8 \text{ m/s.}
\]

(d) Since \( v = H_0d \), then \( H_0 = \frac{v}{d} \). The distance is 4000 Mpc. Just plugging this in gives \( H_0 = 2.88 \times 10^8 \text{km/s}/4000 \text{ Mpc} \) or 72 km/s/Mpc, which is the commonly quoted value. Converting this to inverse seconds gives

\[
H_0 = \frac{v}{d} = \frac{2.88 \times 10^8}{4000 \times 31 \times 10^{21}} = 2.32 \times 10^{-15} \text{ s}^{-1}.
\]

(e) Using our result from part (d) gives

\[
t_0 = \frac{1}{H_0} = \frac{1}{2.88 \times 10^{-15} \text{ s}^{-1}} = 4.31 \times 10^{14} \text{ s}.
\]

Converting this to years gives 13.7 billion years. The current best estimate on the age of the universe is 13.72 \pm 0.12 billion years, and is found using methods like these!
6. **Gravitational Redshift.** *This problem is extra credit!*

Suppose we have a rocket of height $h$ with a laser at the bottom, pointing up. The laser fires a series of pulses, separated by time intervals $T$. At the top of the rocket we have an observer counting the pulses. The rocket is accelerating upwards at a rate $g$, which is the usual gravitational constant. We expect that the observer won’t measure the same time between the pulses as the laser is sending out since the observer is “running away from the light,” as she accelerates. Let’s figure out what she sees.

(a) Show that it takes a time $t_1$ for the observer to receive the first pulse, and $t_2$ to receive the second pulse, where

$$t_1 = \frac{c}{g} \left[ 1 - \sqrt{1 - \frac{2gh}{c^2}} \right],$$

$$t_2 = \frac{c}{g} \left[ 1 - \sqrt{1 - \frac{2g(h+cT)}{c^2}} \right].$$

*Hint: You’ll get a quadratic equation that you need to solve; argue that you need to take the negative sign in the solution.*

(b) The period that the observer measures is just $T' = t_2 - t_1$. Using the binomial expansion, noting that both $2gh \ll c^2$ and $2g(h+cT) \ll c^2$, expand your result to first order in $T$ (i.e., dropping all terms $T^2$ and higher) to find

$$T' = \left( 1 + \frac{gh}{c^2} \right) T.$$

(c) Invert your result for the period to find the observed *frequency*, $f_{\text{obs}}$, in terms of the emitted frequency of the light, $f_{\text{source}}$. Noting again that $gh \ll c^2$, show that

$$f_{\text{obs}} = \left( 1 - \frac{gh}{c^2} \right) f_{\text{source}}.$$

This expression contains a surprising result. The frequency depends on the acceleration, which we have chosen to be $a = g$, the acceleration due to gravity on Earth. If we ignore the derivation of this equation, and just look at the result, then we would be led to believe that a beam of light just traveling upwards in a gravitational field would lose frequency! This is, in fact, completely true. Light is affected by gravity, and as light tries to escape from a gravitational field it experiences a *redshift*, causing its frequency to decrease (hence becoming redder). We can think of this in another (classical) way. When we throw a ball up into the air, it slows down, using its kinetic energy to do work against the force of gravity. Light has to do work against gravity, too, but it can’t change it speed. Therefore it has to lose energy, not by losing speed, but by losing frequency, since the energy of light depends on its frequency. This follows directly from Einstein’s Theory of General Relativity, which says that acceleration and gravity are equivalent.
Solution

(a) Suppose that the rocket starts from rest. Then when the laser fires it reaches the top of the rocket in a time $t_1$. In that time the light travels a distance $y_1 = ct_1$, which is the initial height, $h$, plus the distance that the rocket travelled during that time, $\frac{1}{2}gt_1^2$, so $y_1 = h + \frac{1}{2}gt_1^2 = ct_1$. The laser sends out the next pulse at a time $T$ later, and the light pulse now reaches the top at a time $t_2$, which is again the initial height, plus the distance the rocket moved during that time, $y_2 = h + \frac{1}{2}gt_2^2$, but the light has travelled only for a time $t_2 - T$, there was no light before the beam was sent out. So, $y_2 = h + \frac{1}{2}gt_2^2 = c(t_2 - T)$. Solving these expressions gives the time it takes for the light to reach the top as

$$
t_1 = \frac{c}{g} \left[ 1 - \sqrt{1 - \frac{2gh}{c^2}} \right],
$$

$$
t_2 = \frac{c}{g} \left[ 1 - \sqrt{1 - \frac{2g(h + cT)}{c^2}} \right],
$$

where we have taken the minus sign in the quadratic equation solution to get the correct $g \to 0$ limit.

(b) Now, the laser sent out the light pulses with period $T$, but our observer sees them with period $T' = t_2 - t_1$, which is

$$
T' = \frac{c}{g} \left[ \sqrt{1 - \frac{2gh}{c^2}} - \sqrt{1 - \frac{2g(h + cT)}{c^2}} \right].
$$

In general we don’t expect the periods to agree. To see what the difference is, let’s expand this result to second order in a Taylor series. In general, both $2gh$ and $2g(h + cT) \ll c^2$, and so we can use the binomial theorem to write

$$
T' \approx \frac{c}{g} \left[ 1 - \frac{gh}{c^2} - \frac{g^2h^2}{2c^4} - 1 + \frac{g(h + cT)}{c^2} + \frac{g^2(h + cT)^2}{2c^4} \right],
$$

Canceling off the common terms and dropping the terms of order $T^2$ gives

$$
T' = \left( 1 + \frac{gh}{c^2} \right) T,
$$

which says that the period of reception of the pulses is longer than the period of emission, as we should expect.

(c) Instead of expressing our result in terms of the period, we can instead express it in terms of the frequency, $f = 1/T$. Calling $f_{\text{obs}}$ the observed frequency, and $f_{\text{source}}$ the emitted frequency, we have

$$
f_{\text{obs}} = \frac{f_{\text{source}}}{1 + \frac{gh}{c^2}}.
$$
Now, in general $gh \ll c^2$, and so we can expand the result once again to find

$$f_{\text{obs}} = \left(1 - \frac{gh}{c^2}\right) f_{\text{source}}.$$

Thus, we find that the acceleration produces a Doppler shift in the frequency (in fact this result could have been obtained from the expression for the Doppler Shift of light,

$$f_{\text{obs}} = \sqrt{\frac{c + v}{c - v}} f_{\text{source}},$$

by writing $v = gt$, where $t = h/c$ is the time to reach the top of the rocket, and expanding the square root recalling that $v \ll c$ - you should check this on your own).

As a final comment, notice that the gravitational potential energy of a mass near the surface of the Earth is $PE_g = mgh$, and so $gh = PE_g/m$, but we know that $PE_g/m \equiv \Phi$, the gravitational potential. Thus, we can write the expression for the frequency as

$$f_{\text{obs}} = \left(1 - \frac{\Phi}{c^2}\right) f_{\text{source}}.$$

This turns out to be a very useful form in the discussion of Einstein’s Theory of General Relativity.