Physics 18 Spring 2011
Homework 5 - Solutions
Wednesday February 16, 2011

Make sure your name is on your homework, and please box your final answer. Because we will be giving partial credit, be sure to attempt all the problems, even if you don’t finish them. The homework is due at the beginning of class on Wednesday, February 23rd. Because the solutions will be posted immediately after class, no late homeworks can be accepted! You are welcome to ask questions during the discussion session or during office hours.

1. Evaluate the dot product \( \vec{A} \cdot \vec{B} \) if

(a) \( \vec{A} = 4\hat{i} + 2\hat{j} \) and \( \vec{B} = -3\hat{i} - 2\hat{j} \)

(b) \( \vec{A} = -4\hat{i} + 2\hat{j} \) and \( \vec{B} = -\hat{i} - 2\hat{j} \)

What are the angles between the vectors in each case?

Solution

Recall that, if \( \vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \), and \( \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k} \), then the dot product is

\[ \vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z. \]

So, we just have to plug in the vectors in each case (noting that \( A_z = B_z = 0 \) here) in each case.

(a) Here, \( \vec{A} \cdot \vec{B} = (4)(-3) + (2)(-2) = -12 - 4 = -16. \)

(b) Here, \( \vec{A} \cdot \vec{B} = (-4)(-1) + (2)(-2) = 4 - 4 = 0. \)

In terms of the angle, \( \theta \), between the vectors, we also know that \( \vec{A} \cdot \vec{B} = AB \cos \theta \), where \( A = \sqrt{A_x^2 + A_y^2 + A_z^2} \) is the magnitude of \( \vec{A} \), and similarly for \( B \). Now, for part (a), \( A = \sqrt{4^2 + 2^2} = \sqrt{20} \), while \( B = \sqrt{3^2 + 2^2} = \sqrt{13} \). Since \( \vec{A} \cdot \vec{B} = -16 \), we have

\[ \theta = \cos^{-1} \left( \frac{-16}{\sqrt{20}\sqrt{13}} \right) \approx 173^\circ. \]

For part (b) our job is much easier. Since the dot product is zero, this means that the two vectors are perpendicular, and so the angle between them is 90°.
2. You run a race with a friend. At first you each have the same kinetic energy, but she is running faster than you are. When you increase your speed by 25 percent, you are running at the same speed she is. If your mass is 85 kg, what is her mass?

Solution

You and your friend both start off with the same kinetic energy, so $KE_{you} = KE_{friend}$. When you increase your speed by 25%, then you both are running at the same speed; if your original speed was $v_{you}$, then $1.25v_{you} = v_{friend}$. So, $v_{you} = \frac{1}{1.25}v_{friend} = \frac{4}{5}v_{friend}$. Thus, since the kinetic energies were originally equal we can solve for her mass

$$\frac{1}{2}m_{you}v_{you}^2 = \frac{1}{2}m_{friend}v_{friend}^2 \Rightarrow m_{friend} = \left(\frac{v_{you}}{v_{friend}}\right)^2 m_{you}.$$ 

Now, plugging in $v_{you} = \frac{4}{5}v_{friend}$, and $m_{you} = 85$ kg, then

$$m_{friend} = \left(\frac{v_{you}}{v_{friend}}\right)^2 m_{you} = \left(\frac{4}{5}\right)^2 \times 85 = 54 \text{ kg}.$$
3. A 5.0 kg cat leaps from the floor to the top of a 95 centimeter high table. If the cat pushes against the floor for 0.20 seconds to accomplish this feat, what was her average power output during the pushoff period?

Solution

The power is just the work done, divided by the time that it takes to do the work. The cat does work against gravity, \( W = mgh \) leaping up to the table, where \( m \) is her mass, \( g \) is the acceleration due to gravity, and \( h \) is the height of the table. It takes her \( T = 0.20 \) seconds to do the work to jump up to the table, so

\[
P = \frac{W}{T} = \frac{mgh}{T} = \frac{5.0 \times 9.8 \times 0.95}{0.20} = 233 \text{ Watts}.
\]
4. You have a vacation cabin that has a nearby solar (black) water container used to provide a warm outdoor shower. For a few days last summer, your pump went out and you had to personally haul the water up the 4.0 m from the pond to the tank. Suppose your bucket has a mass of 5.0 kg and holds 15.0 kg of water when it is full. However, the bucket has a hole in it, and as you moved it vertically at a constant speed $v$, water leaked out at a constant rate. By the time you reached the top, only 5.0 kg of water remained.

(a) Write an expression for the mass of the bucket plus water as a function of the height above the pond surface.

(b) Find the work done by you on the bucket for each 5.0 kg of water delivered to the tank.

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**Solution**

(a) If the bucket was full, it would be 20 kg (the bucket, plus the water). As you walk along back to the shower, the water drains out at a constant rate. So, the total mass of the bucket would be 20 kg, minus the amount that has drained out. How much has drained out? We’re told that the rate at which the bucket drains is constant, which means that rate $r = \frac{\Delta m}{\Delta y} = \text{constant}$. If the bucket loses 10 kg of water over a change of 4 meters, then $r = \frac{10}{4} = 2.5$ kg/m. So the change in mass at some height $y$ is just $\Delta m = ry$. Thus, the total mass is

$$m(y) = 20 - ry = 20 - 2.5y.$$  

(b) The work done is just $W = \int Fdy$, since we’re moving along the $y$ direction. The force is the gravitational force $F = mg$, but the mass changes as you walk up the path. So, the total work is

$$W = \int_0^4 Fdy = g \int_0^4 (20 - 2.5y) dy = g \left[(20y - 1.25y^2)\right]_0^4 = 9.8 (20 \times 4 - 1.5 \times 16) = 588 \text{ J}.$$
5. Particle \( a \) has mass \( m \), is initially located on the positive \( x \) axis at \( x = x_0 \) and is subject to a repulsive force \( F_x \) from particle \( b \). The location of particle \( b \) is fixed at the origin. The force \( F_x \) is inversely proportional to the square of the distance \( x \) between the particles. That is, \( F_x = A/x^2 \), where \( A \) is a positive constant. Particle \( a \) is released from rest and allowed to move under the influence of the force. Find an expression for the work done by the force on \( a \) as a function of \( x \). Find both the kinetic energy and speed of \( a \) as \( x \) approaches infinity.

Solution

Particle \( a \) moves under the influence of a particle moving entirely along \( x \), and so the work is just \( W = \int Fdx \), since \( \cos \theta = 1 \) for a particle moving along the direction of \( F \). So, the work done by the force moving the particle from some initial point \( x_0 \) to a final position \( x \) is

\[
W = \int_{x_0}^{x} Fdx \\
= A \int_{x_0}^{x} \frac{dx}{x^2} \\
= \left[ -\frac{A}{x} \right]_{x_0}^{x} \\
= A \left( \frac{1}{x} - \frac{1}{x_0} \right).
\]

Thus, we find the work done

\[
W(x) = A \left( \frac{1}{x} - \frac{1}{x_0} \right).
\]

The work done is just the change in kinetic energy, \( W = \Delta KE \). So, since the particle is released from rest, \( KE_i = 0 \), and so \( \Delta KE = KE \), and

\[
KE = A \left( \frac{1}{x} - \frac{1}{x_0} \right).
\]

As \( x \to \infty \) (i.e., as the particles move away from each other), then the second term in the parenthesis goes to zero, and so

\[
KE = \frac{A}{x_0}.
\]

Since \( KE = \frac{1}{2}mv^2 \), we can easily solve for the velocity to find

\[
v = \sqrt{\frac{2A}{mx_0}}.
\]
6. Very soon we shall introduce a new vector for a particle, called its *linear momentum*, symbolized by \( \vec{p} \). Mathematically, it is related to the mass \( m \) and velocity \( \vec{v} \) of the particle by \( \vec{p} = m\vec{v} \).

(a) Show that the particle’s kinetic energy \( K \) can be expressed as \( K = \frac{\vec{p} \cdot \vec{p}}{2m} \).

(b) Compute the linear momentum of a particle of mass 2.5 kg that is moving at a speed of 15 m/s at an angle of 25° clockwise from the +x axis in the xy plane.

(c) Compute its kinetic energy using both \( K = \frac{1}{2}mv^2 \) and \( K = \frac{\vec{p} \cdot \vec{p}}{2m} \) and verify that they give the same result.

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**Solution**

(a) The momentum is \( \vec{p} = m\vec{v} \). Now, \( \vec{p} \cdot \vec{p} = m^2\vec{v} \cdot \vec{v} \). But \( \vec{v} \cdot \vec{v} = v^2 \), as is true for the dot product of any vector with itself, since the angle between a vector and itself is zero. Now, the kinetic energy is \( KE = \frac{1}{2}mv^2 \), and we can substitute in for \( v^2 = \vec{p} \cdot \vec{p} / m^2 \) to find

\[
KE = \frac{1}{2}mv^2 = \frac{m \vec{p} \cdot \vec{p}}{2m^2} = \frac{\vec{p} \cdot \vec{p}}{2m}.
\]

(b) The velocity of the particle is \( \vec{v} = v_x \hat{i} + v_y \hat{j} = v_0 \cos \theta \hat{i} + v_0 \sin \theta \hat{j} \). So, \( \vec{v} = 15 \cos 25^\circ \hat{i} + 15 \sin 25^\circ \hat{j} = 13.6 \hat{i} + 6.3 \hat{j} \). The momentum is \( \vec{p} = m\vec{v} = 2.5 \times (13.6 \hat{i} + 6.3 \hat{j}) \), or

\[
\vec{p} = 34 \hat{i} + 15.8 \hat{j} \text{ kg m/s}.
\]

(c) We can work out the kinetic energy two ways:

\[
KE = \frac{1}{2}mv^2 = \frac{1}{2}(2.5)(15)^2 = 281 \text{ J},
\]

while

\[
KE = \frac{\vec{p} \cdot \vec{p}}{2m} = \frac{(34)^2 + (15.8)^2}{2 \times 2.5} = 281 \text{ J}.
\]

So, the two methods agree.
7. A particle moves in a circle that is centered at the origin and the magnitude of its position vector $\vec{r}$ is constant.

(a) Differentiate $\vec{r} \cdot \vec{r} = r^2 = \text{constant}$ with respect to time to show that $\vec{v} \cdot \vec{r} = 0$, and therefore $\vec{v} \perp \vec{r}$.

(b) Differentiate $\vec{v} \cdot \vec{r} = 0$ with respect to time and show that $\vec{a} \cdot \vec{r} + v^2 = 0$, and therefore $a_r = -v^2/r$.

(c) Differentiate $\vec{v} \cdot \vec{v} = v^2$ with respect to time to show that $\vec{a} \cdot \vec{v} = dv/dt$, and therefore $a_t = dv/dt$.

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**Solution**

(a) Let’s differentiate $\vec{r} \cdot \vec{r} = r^2$, where $r$ is a constant, with respect to time. Recalling that for any dot product, \[
\frac{d}{dt} (\vec{A} \cdot \vec{B}) = \frac{d\vec{A}}{dt} \cdot \vec{B} + \vec{A} \cdot \frac{d\vec{B}}{dt},
\] we find
\[
\frac{d}{dt} (\vec{r} \cdot \vec{r}) = \dot{\vec{r}} \cdot \vec{r} + \vec{r} \cdot \dot{\vec{r}} = 2\dot{\vec{r}} \cdot \vec{r},
\]
where we recall that $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$. Now $\frac{d}{dt} (\vec{r} \cdot \vec{r}) = \frac{d}{dt} (r^2) = 0$, since $r$ is a constant (since the particle is moving in a circle). So, we see that, since $\vec{v} = \dot{\vec{r}}$, $\vec{v} \cdot \vec{r} = 0$. Since $\vec{r}$ and $\vec{v}$ are vectors, and neither is zero, we see from the dot product $\vec{v} \cdot \vec{r} = vr \cos \theta$, that $\vec{r}$ and $\vec{v}$ must be perpendicular.

(b) Now, differentiating $\frac{d}{dt} (\vec{v} \cdot \vec{r})$ gives
\[
\frac{d}{dt} (\vec{v} \cdot \vec{r}) = \dot{\vec{v}} \cdot \vec{r} + \vec{v} \cdot \dot{\vec{r}} = \vec{a} \cdot \vec{r} + v^2.
\]
Now, since $\vec{v} \cdot \vec{r} = 0$, then $\vec{a} \cdot \vec{r} + v^2 = 0$. Now, $\vec{a} \cdot \vec{r}$ gives the component of the acceleration along $\vec{r}$, and so $\vec{a} \cdot \vec{r} = a_r r$. Thus, solving for $a_r$ gives
\[
a_r = -\frac{v^2}{r},
\]
which is just the centripetal acceleration.

(c) Now, differentiating gives
\[
\frac{d}{dt} (\vec{v} \cdot \vec{v}) = 2\vec{a} \cdot \vec{v}.
\]
But, $\vec{v} \cdot \vec{v} = v^2$, so $\frac{d}{dt} (\vec{v} \cdot \vec{v}) = \frac{d}{dt} (v^2) = 2v \frac{dv}{dt}$. So, comparing gives $\vec{a} \cdot \vec{v} = v \frac{dv}{dt}$. Dividing by the velocity and recalling that the unit vector $\hat{v} = \vec{v}/v$, we find
\[
\vec{a} \cdot \hat{v} = \frac{dv}{dt}.
\]
Now, $\vec{a} \cdot \hat{v}$ gives the component of acceleration in the direction of the velocity; in other words it gives the tangential acceleration,
\[
a_t = \frac{dv}{dt}.
\]
8. Estimate the maximum speed of a horse. Assume that the horse is 1.8 meters tall and 0.5 meters wide. *Hint:* recall that 1 “horsepower” is 746 watts. Furthermore, remember that the horse is also subject to a resistive drag force, \( F_{\text{drag}} \approx \frac{1}{4} Av^2 \), where \( A \) is the area of the horse.

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**Solution**

The horse generates a power of one horsepower, by definition. When he’s running at a constant speed, then the net force is zero - the horse is generating a force opposing the drag force, so \( F_{\text{horse}} = F_{\text{drag}} \approx \frac{1}{4} Av^2 \). The power is \( P_{\text{horse}} = F_{\text{horse}}v \approx \frac{1}{4} Av^3 \). Solving for the velocity gives

\[
v \approx \left( \frac{4P}{A} \right)^{1/3} = \left( \frac{4 \times 746}{1.8 \times 0.5} \right)^{1/3} \approx 14.9 \text{ m/s},
\]

which is about 33 miles per hour, and is a completely reasonable speed.
9. A box of mass $M$ is at rest at the bottom of a frictionless inclined plane. The box is attached to a string that pulls with a constant tension $T$.

(a) Find the work done by the tension $T$ as the box moves through a distance $x$ along the plane.

(b) Find the speed of the box as a function of $x$.

(c) Determine the power delivered by the tension in the string as a function of $x$.

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**Solution**

(a) The work done by the tension as the box moves along the plane is just the force (tension) times the distance it moves, $x$. So, $W = Tx$.

(b) We’ll figure out the speed using the work done on the block, since $W = \Delta KE$. The net force acting along the $x$ direction is just $F_{\text{net}} = T - Mg\sin\theta$, as we can see from the figure to the right. This force is constant. So, the net work on the block, if it moves through a distance $x$ is $W = F_{\text{net}}x = (T - Mg\sin\theta)x$. Since the block starts from rest, $KE_i = 0$, and so $W = KE = \frac{1}{2}Mv^2$. Solving for the velocity gives

$$v = \sqrt{2 \left(\frac{T}{M} - g\sin\theta\right)x}.$$  

(c) Now, the power delivered to the block by the tension is $P = \vec{T} \cdot \vec{v} = Tv$, since both are in the same direction. So, using our answer from part (b) gives

$$P = Tv = T\sqrt{2 \left(\frac{T}{M} - g\sin\theta\right)x}.$$
10. A force acting on a particle in the \(xy\) plane at coordinates \((x, y)\) is given by \(\vec{F} = (F_0/r) \left( y\hat{i} - x\hat{j} \right)\), where \(F_0\) is a positive constant and \(r\) is the distance of the particle from the origin.

(a) Show that the magnitude of this force is \(F_0\).

(b) Show that the direction of \(\vec{F}\) is perpendicular to \(\vec{r} = x\hat{i} + y\hat{j}\).

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**Solution**

(a) The magnitude of the force is 
\[
|\vec{F}| = \sqrt{F_x^2 + F_y^2} = \sqrt{(F_0/r)^2 y^2 + (F_0/r)^2 x^2} = F_0 \sqrt{\frac{x^2 + y^2}{r^2}} = F_0, 
\]
where we have recalled that \(r = \sqrt{x^2 + y^2}\).

(b) We can see that the direction of \(\vec{F}\) is perpendicular to \(\vec{r}\) by taking the dot product \(\vec{F} \cdot \vec{r}\). Doing so gives
\[
\vec{F} \cdot \vec{r} = F_x r_x + F_y r_y = \frac{F_0}{r} (yx - xy) = 0.
\]

The dot product of \(\vec{r}\) and \(\vec{F}\) is zero, and so they are perpendicular.