7. (a) Consider the mapping \( w = z^3 \). When we encircle the origin in the \( z \) plane one time, how many times do we encircle the origin in the \( w \) plane? Explain why this agrees with the Argument Principle.

(b) Suppose we consider \( w = z^3 + a_2 z^2 + a_1 z + a_0 \) for three constants \( a_0, a_1, a_2 \). If we encircle the origin in the \( z \) plane once on a very large circle, how many times do we encircle the \( w \) plane?

(c) Suppose we have a mapping \( w = f(z) \) where \( f(z) \) is analytic inside and on a simple closed contour \( C \) in the \( z \) plane. Let us define \( \tilde{C} \) as the (nonsimple) image in the \( w \) plane of the contour \( C \) in the \( z \) plane. If we deduce that it encloses the origin \( (w = 0) \) \( N \) times, and encloses the point \( (w = 1) \) \( M \) times, what is the change in \( \arg w \) over the contour \( \tilde{C} \)? If we had a computer available, what algorithm should be designed (if it is at all possible) to determine the change in the argument?

*4.5 Fourier and Laplace Transforms*

One of the most valuable tools in mathematics, physics, and engineering is making use of the properties a function takes on in a so-called transform (or dual) space. In suitable function spaces, defined below, the Fourier transform pair is given by the following relations:

\[
\begin{align*}
\hat{F}(k) &= \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx & (4.5.2) \\
F(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(k) e^{ikx} \, dk & (4.5.1)
\end{align*}
\]

\( \hat{F}(k) \) is called the Fourier transform of \( f(x) \). The integral in Eq. (4.5.1) is referred to as the inverse Fourier transform. In mathematics, the study of Fourier transforms is central in fields like harmonic analysis. In physics and engineering, applications of Fourier transforms are crucial, for example, in the study of quantum mechanics, wave propagation, and signal processing. In this section we introduce the basic notions and give a heuristic derivation of Eqs. (4.5.1)–(4.5.2). In the next section we apply these concepts to solve some of the classical partial differential equations.

In what follows we make some general remarks about the relevant function spaces for \( f(x) \) and \( \hat{F}(k) \). However, in our calculations we will apply complex variable techniques and will not use any deep knowledge of function spaces. Relations (4.5.1)–(4.5.2) always hold if \( f(x) \in L^2 \) and \( \hat{F}(k) \in L^2 \), where \( f \in L^2 \) refers to the function space of square integrable functions

\[
\| f \|_2 = \left( \int_{-\infty}^{\infty} |f|^2(x) \, dx \right)^{1/2} < \infty \tag{4.5.3}
\]
In those cases where \( f(x) \) and \( f'(x) \) are continuous everywhere apart from a finite number of points for which \( f(x) \) has integrable and bounded discontinuities (such a function \( f(x) \) is said to be \textbf{piecewise smooth}), it turns out that at each point of discontinuity, call it \( x_0 \), the integral given by Eq. (4.5.1) converges to the mean: \( \lim_{\epsilon \to 0^+} \frac{1}{2} [f(x_0 + \epsilon) + f(x_0 - \epsilon)] \). At points where \( f(x) \) is continuous, the integral (4.5.1) converges to \( f(x) \).

There are other function spaces for which Eqs. (4.5.1)–(4.5.2) hold. If \( f(x) \in L^1 \), the space of \textbf{absolutely integrable functions} (functions satisfying \( \int_{-\infty}^{\infty} |f(x)| \, dx < \infty \)), then \( \hat{F}(k) \) tends to zero for \( |k| \to \infty \) and belongs to a certain function space of functions decaying at infinity (sometimes referred to as \textbf{bounded mean oscillations} (BMO)). Conversely, if we start with a suitably decaying function \( \hat{F}(k) \) (i.e., in BMO), then \( f(x) \in L^1 \). In a sense, purely from a symmetry point of view, such function spaces may seem less natural than when \( f(x) \), \( \hat{F}(k) \) are both in \( L^2 \) for Eqs. (4.5.1)–(4.5.2) to be valid. Nevertheless, in some applications \( f(x) \in L^1 \) and not \( L^2 \). Applications sometimes require the use of Fourier transforms in spaces for which no general theory applies, but nevertheless, specific results can be attained. It is outside the scope of this text to examine \( L^p \) \((p = 1, 2, \ldots)\) function spaces where the general results pertaining to Eqs. (4.5.1)–(4.5.2) can be proven. Interested readers can find such a discussion in various books on complex or Fourier analysis, such as Rudin (1966). Unless otherwise specified, we shall assume that our function \( f(x) \in L^1 \cap L^2 \), that is, \( f(x) \) is both absolutely and square integrable. It follows that

\[
|\hat{F}(k)| \leq \|f\|_1 = \int_{-\infty}^{\infty} |f(x)| \, dx
\]

and

\[
\frac{1}{\sqrt{2\pi}} \|\hat{F}\|_2 = \|f\|_2
\]

The first relationship follows directly. The second is established later in this section. In those cases for which \( f(x) \) is piecewise smooth, an elementary though tedious proof of Eqs. (4.5.1)–(4.5.2) can be constructed by suitably breaking up the interval \((-\infty, \infty)\) and using standard results of integration (Titchmarsh, 1948).

Statements analogous to Eqs. (4.5.1)–(4.5.2) hold for functions on finite intervals \((-L, L)\), which may be extended as periodic functions of period \(2L\):
The values $\hat{F}_n$ are called the **Fourier coefficients** of the Fourier series representation of the function $f(x)$ given by Eq. (4.5.4).

A simple example of how Fourier transforms may be calculated is afforded by the following.

**Example 4.5.1** Let $a > 0$, $b > 0$, and $f(x)$ be given by

$$f(x) = \begin{cases} e^{-ax} & x > 0 \\ e^{bx} & x < 0 \end{cases}$$

Then, from Eq. (4.5.2), we may compute $\hat{F}(k)$ to be

$$\hat{F}(k) = \int_{0}^{\infty} e^{-ax - ikx} \, dx + \int_{-\infty}^{0} e^{bx - ikx} \, dx$$

$$= \frac{1}{a + ik} + \frac{1}{b - ik} \quad (4.5.6)$$

The inversion (i.e., reconstruction of $f(x)$) via Eq. (4.5.1) is ascertained by calculating the appropriate contour integrals. In this case we use closed semicircles in the upper ($x > 0$) and lower ($x < 0$) half $k$-planes. The inversion can be done either by combining the two terms in Eq. (4.5.6) or by noting that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{a + ik} \, dk = \begin{cases} e^{-ax} & x > 0 \\ 1/2 & x = 0 \\ 0 & x < 0 \end{cases}$$

and

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{b - ik} \, dk = \begin{cases} 0 & x > 0 \\ 1/2 & x = 0 \\ e^{bx} & x < 0 \end{cases}$$

The values at $x = 0$ take into account both the pole and the principal value contribution at infinity; that is, $\frac{1}{2\pi} \int_{C_{R}} \frac{dk}{a + ik} = 1/2$ where on $C_{R} : k = R e^{i\theta}$, $0 \leq \theta \leq \pi$. Thus Eq. (4.5.1) gives convergence to the mean value at $x = 0$: $\frac{1}{2} + \frac{1}{2} = 1 = (\lim_{x \to 0^+} f(x) + \lim_{x \to 0^-} f(x))/2$.

A function that will lead us to a useful result is the following:

$$\Delta(x; \epsilon) = \begin{cases} 1/2\epsilon & |x| < \epsilon \\ 0 & |x| > \epsilon \end{cases} \quad (4.5.7)$$

Its Fourier transform is given by

$$\hat{\Delta}(k; \epsilon) = \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} e^{-ikx} \, dx = \frac{\sin k\epsilon}{k\epsilon} \quad (4.5.8)$$
Certainly, $\Delta(x; \epsilon)$ is both absolutely and square integrable; so it is in $L^1 \cap L^2$, and $\hat{\Delta}(k; \epsilon)$ is in $L^2$. It is natural to ask what happens as $\epsilon \to 0$. The function defined by Eq. (4.5.7) tends, as $\epsilon \to 0$, to a novel “function” called the **Dirac delta function**, denoted by $\delta(x)$ and having the following properties:

$$\delta(x) = \lim_{\epsilon \to 0} \Delta(x; \epsilon) \quad (4.5.9)$$

$$\int_{-\infty}^{\infty} \delta(x - x_0) \, dx = \lim_{\epsilon \to 0} \int_{x=x_0-\epsilon}^{x=x_0+\epsilon} \delta(x - x_0) \, dx = 1 \quad (4.5.10)$$

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) \, dx = f(x_0) \quad (4.5.11)$$

where $f(x)$ is continuous. Equations (4.5.10)–(4.5.11) can be ascertained by using the limit definitions (4.5.7) and (4.5.9). The function defined in Eq. (4.5.9) is often called a unit impulse “function”; it has an arbitrarily large value concentrated at the origin, whose integral is unity. The delta function, $\delta(x)$, is not a mathematical function in the conventional sense, as it has an arbitrarily large value at the origin. Nevertheless, there is a rigorous mathematical framework in which these new functions – called distributions – can be analyzed. Interested readers can find such a discussion in, for example, Lighthill (1959). For our purposes the device of the limit process $\epsilon \to 0$ is sufficient. We also note that Eq. (4.5.7) is not the only valid representation of a delta function; for example, others are given by

$$\delta(x) = \lim_{\epsilon \to 0} \left\{ \frac{1}{\sqrt{\pi \epsilon}} e^{-x^2/\epsilon} \right\} \quad (4.5.12a)$$

$$\delta(x) = \lim_{\epsilon \to 0} \left\{ \frac{\epsilon}{\pi (\epsilon^2 + x^2)} \right\} \quad (4.5.12b)$$

It should be noted that, formally speaking, the Fourier transform of a delta function is given by

$$\hat{\delta}(k) = \int_{-\infty}^{\infty} \delta(x) e^{-ikx} \, dx = 1 \quad (4.5.13)$$

It does not vanish as $|k| \to \infty$; indeed, it is a constant (unity; note that $\delta(x)$ is not $L^1$). Similarly, it turns out from the theory of distributions (motivated by the inverse Fourier transform), that the following alternative definition of a delta function holds:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\delta}(k) e^{ikx} \, dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, dk \quad (4.5.14)$$
4.5 Fourier and Laplace Transforms

Formula (4.5.14) allows us a simple (but formal) way to verify Eqs. (4.5.1)–(4.5.2). Namely, by using Eqs. (4.5.1)–(4.5.2) and assuming that interchanging integrals is valid, we have

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{ikx} \left( \int_{-\infty}^{\infty} dx' \, f(x') \, e^{-ikx'} \right)
\]

\[
= \int_{-\infty}^{\infty} dx' \, f(x') \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{ik(x-x')} \right)
\]

\[
= \int_{-\infty}^{\infty} dx' \, f(x') \delta(x - x')
\]

(4.5.15)

In what follows, the Fourier transform of derivatives will be needed. The Fourier transform of a derivative is readily obtained via integration by parts.

\[
\hat{F}_1(k) \equiv \int_{-\infty}^{\infty} f'(x) \, e^{-ikx} \, dx
\]

\[
= \left[ f(x) \, e^{-ikx} \right]_{-\infty}^{\infty} + ik \int_{-\infty}^{\infty} f(x) \, e^{-ikx} \, dx
\]

\[
= ik \, \hat{F}(k)
\]

(4.5.16a)

and by repeated integration by parts

\[
\hat{F}_n(k) = \int_{-\infty}^{\infty} f^{(n)}(x) e^{-ikx} \, dx = (ik)^n \hat{F}(k)
\]

(4.5.16b)

Formulae (4.5.16a,b) will be useful when we examine solutions of differential equations by transform methods.

It is natural to ask what is the Fourier transform of a product. The result is called the convolution product; it is not the product of the Fourier transforms. We can readily derive this formula. We use two transform pairs: one for a function \( f(x) \) (Eqs. (4.5.1)–(4.5.2)) and another for a function \( g(x) \), replacing \( f(x) \) and \( \hat{F}(k) \) in Eqs. (4.5.1)–(4.5.2) by \( g(x) \) and \( \hat{G}(k) \), respectively. We define the convolution product,

\[
(f * g)(x) = \int_{-\infty}^{\infty} f(x - x')g(x') \, dx' = \int_{-\infty}^{\infty} f(x')g(x - x') \, dx'.
\]

(The latter equality follows by renaming variables.) We take the Fourier transform of \((f * g)(x)\) and interchange integrals (allowed since both \( f \) and \( g \) are
absolutely integrable) to find
\[
\int_{-\infty}^{\infty} (f * g)(x)e^{-ikx} \, dx = \int_{-\infty}^{\infty} dx \left( \int_{-\infty}^{\infty} f(x - x')g(x')dx' \right) e^{-ikx}
\]
\[
= \int_{-\infty}^{\infty} dx' e^{ikx'}g(x') \int_{-\infty}^{\infty} dx e^{-ik(x-x')} f(x - x')
\]
\[= \hat{F}(k)\hat{G}(k). \tag{4.5.17}\]

Hence, by taking the inverse transform of this result,
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{F}(k)\hat{G}(k) \, dk = \int_{-\infty}^{\infty} g(x') f(x - x') \, dx'
\]
\[= \int_{-\infty}^{\infty} g(x - x') f(x')dx'. \tag{4.5.18}\]

The latter equality is accomplished by renaming the integration variables. Note that if \( f(x) = \delta(x - x') \), then \( \hat{F}(k) = 1 \), and Eq. (4.5.18) reduces to the known transform pair for \( g(x) \), that is, \( g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}(k)e^{ikx} \, dk \).

A special case of Eq. (4.5.18) is the so-called Parseval formula, obtained by taking \( g(x) = \overline{f}(-x) \), where \( \overline{f}(x) \) is the complex conjugate of \( f(x) \), and evaluating Eq. (4.5.17) at \( x = 0 \):
\[
\int_{-\infty}^{\infty} f(x)\overline{f}(x) \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(k)\hat{G}(k) \, dk.
\]

Function \( \hat{G}(k) \) is now the Fourier transform of \( \overline{f}(-x) \):
\[
\hat{G}(k) = \int_{-\infty}^{\infty} e^{-ikx} \overline{f}(-x) \, dx = \int_{-\infty}^{\infty} e^{ikx} \overline{f}(x) \, dx
\]
\[= \left( \int_{-\infty}^{\infty} e^{-ikx} f(x) \, dx \right) = (\hat{F}(k))
\]

Hence we have the Parseval formula:
\[
\int_{-\infty}^{\infty} |f|^2(x) \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{F}(k)|^2 \, dk \tag{4.5.19}
\]

In some applications, \( \int_{-\infty}^{\infty} |f|^2(x) \, dx \) refers to the energy of a signal. Frequently it is the term \( \int_{-\infty}^{\infty} |\hat{F}(k)|^2 \, dk \) which is really measured (it is sometimes referred to as the power spectrum), which then gives the energy as per Eq. (4.5.19).
In those cases where we have \( f(x) \) being an even or odd function, then the Fourier transform pair reduces to the so-called **cosine transform** or **sine transform** pair. Because \( f(x) \) being even/odd in \( x \) means that \( \hat{F}(k) \) will be even or odd in \( k \), the pair (4.5.1)–(4.5.2) reduces to statements about semiinfinite functions in the space \( L^2(0, \infty) \) (i.e., \( \int_0^\infty |f(x)|^2 \, dx < \infty \)) or in the space \( L^1(0, \infty) \) (i.e., \( \int_0^\infty |f(x)| \, dx < \infty \)). For even functions, \( f(x) = f(-x) \), the following definitions

\[
f(x) = \frac{1}{\sqrt{2}} f_c(x), \quad \hat{F}(k) = \sqrt{2} \hat{F}_c(k)
\]  

(4.5.20)

(or more generally \( \hat{F}(k) = a \hat{F}_c(k), \, f(x) = b f_c(x), \, a = 2b, \, b \neq 0 \)) yield the Fourier cosine transform pair

\[
f_c(x) = \frac{2}{\pi} \int_0^{\infty} \hat{F}_c(k) \cos kx \, dk
\]

(4.5.21)

\[
\hat{F}_c(k) = \int_0^{\infty} f_c(x) \cos kx \, dx
\]

(4.5.22)

For odd functions, \( f(x) = -f(-x) \), the definitions

\[
f(x) = \frac{1}{\sqrt{2}} f_s(x), \quad \hat{F}(k) = -\sqrt{2}i \hat{F}_s(k)
\]

yield the Fourier sine transform pair

\[
f_s(x) = \frac{2}{\pi} \int_0^{\infty} \hat{F}_s(k) \sin kx \, dk
\]

(4.5.23)

\[
\hat{F}_s(k) = \int_0^{\infty} f_s(x) \sin kx \, dx
\]

(4.5.24)

Obtaining the Fourier sine or cosine transform of a derivative employs integration by parts; for example

\[
\hat{F}_{c,1}(k) = \int_0^{\infty} f'(x) \cos kx \, dk
\]

\[
= [f(x) \cos kx]_0^\infty + k \int_0^{\infty} f(x) \sin kx \, dk = k \hat{F}_s(k) - f(0)
\]

(4.5.25)

and

\[
\hat{F}_{s,1}(k) = \int_0^{\infty} f'(x) \sin kx \, dk
\]

\[
= [f(x) \sin kx]_0^\infty - k \int_0^{\infty} f(x) \cos kx \, dk = -k \hat{F}_c(k)
\]

(4.5.26)

are formulae for the first derivative. Similar results obtain for higher derivatives.
It turns out to be useful to extend the notion of Fourier transforms. One way to do this is to consider functions that have support only on a semi-interval. We take \( f(x) = 0 \) on \( x < 0 \) and replace \( f(x) \) by \( e^{-cx} f(x) \) for \( x > 0 \). Then Eqs. (4.5.1)–(4.5.2) satisfy, using \( \hat{F}(k) \) from Eq. (4.5.2) in Eq. (4.5.1):

\[
e^{-cx} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \left[ \int_{0}^{\infty} e^{-ikx'} e^{-cx'} f(x') dx' \right]
\]

hence

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{(c+ik)x} \left[ \int_{0}^{\infty} e^{-(c+ik)x'} f(x') dx' \right]
\]

Within the above integrals we define \( s = c + ik \), where \( c \) is a fixed real constant, and make the indicated redefinition of the limits of integration to obtain

\[
f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds e^{sx} \left( \int_{0}^{\infty} e^{-sx'} f(x') dx' \right)
\]

or, in a form analogous to Eqs. (4.5.1)–(4.5.2):

\[
f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{F}(s) e^{sx} ds \quad (4.5.27)
\]

\[
\hat{F}(s) = \int_{0}^{\infty} f(x) e^{-sx} dx \quad (4.5.28)
\]

Formulae (4.5.27)–(4.5.28) are referred to as the \textbf{Laplace transform} \( \hat{F}(s) \) and the inverse Laplace transform of a function \( f(x) \), respectively. The usual function space for \( f(x) \) in the Laplace transform (analogous to \( L^1 \cap L^2 \) for \( f(x) \) in Eq. (4.5.1)) are those functions satisfying:

\[
\int_{0}^{\infty} e^{-cx} |f(x)| dx < \infty. \quad (4.5.29)
\]

Note \( \text{Re } s = c \) in Eqs. (4.5.27)–(4.5.28). If Eq. (4.5.29) holds for some \( c > 0 \), then \( f(x) \) is said to be of \textbf{exponential order}.

The integral (4.5.27) is generally carried out by contour integration. The contour from \( c - i\infty \) to \( c + i\infty \) is referred to as the \textbf{Bromwich contour}, and \( c \) is taken to the right of all singularities in order to insure (4.5.29). Closing the contour to the right will yield \( f(x) = 0 \) for \( x < 0 \).
We give two examples.

**Example 4.5.2** Evaluate the inverse Laplace transform of \( \hat{F}(s) = \frac{1}{s^2 + \omega^2} \), with \( \omega > 0 \). See Figure 4.5.1.

\[
f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{sx}}{s^2 + \omega^2} \, ds
\]

For \( x < 0 \) we close the contour to the right of the Bromwich contour. Because no singularities are enclosed, we have, on \( C_{R_2} \)

\[
s = c + Re^{i\theta}, \quad \frac{-\pi}{2} < \theta < \frac{\pi}{2}
\]

thus \( f(x) = 0, x < 0 \) because \( \int_{C_{R_2}} \to 0 \) as \( R \to \infty \). On the other hand, for \( x > 0 \), closing to the left yields (we note that \( \int_{C_{R_1}} \to 0 \) as \( R \to \infty \) where on \( C_{R_1} \):

\[
s = c + Re^{i\theta}, \quad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}
\]

\[
f(x) = \sum_{j=1}^{2} \text{Res} \left( \frac{e^{sx}}{s^2 + \omega^2}; s_j \right), \quad s_1 = iw, \quad s_2 = -iw
\]

\[
f(x) = \frac{e^{iw}x}{2iw} - \frac{e^{-iw}x}{2iw} = \frac{\sin wx}{w} \quad (x > 0)
\]

**Example 4.5.3** Evaluate the inverse Laplace transform of the function

\[
\hat{F}(s) = s^{-a}, \quad (0 < a < 1)
\]

where we take the branch cut along the negative real axis (see Figure 4.5.2).
As is always the case, \( f(x) = 0 \) for \( x < 0 \) when we close to the right. Closing to the left yields (schematically)

\[
\int_{C_R} f(s) e^{sx} ds = 0
\]

The right-hand side vanishes because there are no singularities enclosed by this contour. The integrals \( \int_{C_{R_A}} \), \( \int_{C_{R_B}} \), and \( \int_{C_{e}} \) vanish as \( R \to \infty \) (on \( C_{R_A} : s = c + Re^{i\theta}, \frac{\pi}{2} < \theta < \pi \); on \( C_{R_B} : s = c + Re^{i\theta}, -\pi < \theta < -\frac{\pi}{2} \), and \( \epsilon \to 0 \) (on \( C_{e} : s = \epsilon e^{i\theta}, -\pi < \theta < \pi \)). On \( C_{e} \) we have

\[
\left| \int_{C_{e}} \frac{F(s)}{2\pi i} e^{sx} ds \right| \leq \int_{-\pi}^{\pi} \epsilon^{1-a} \frac{e^{\epsilon x}}{2\pi} d\theta \to 0 \quad \epsilon \to 0
\]

Hence

\[
f(x) - \frac{1}{2\pi i} \int_{-\infty}^{0} r^{-a} e^{-i\alpha x} e^{-rx} dr - \frac{1}{2\pi i} \int_{0}^{\infty} r^{-a} e^{i\alpha x} e^{-rx} dr = 0
\]

or

\[
f(x) = \frac{(e^{i\alpha x} - e^{-i\alpha x})}{2\pi i} \int_{0}^{\infty} r^{-a} e^{-rx} dr \quad (4.5.30)
\]

The **gamma function**, or factorial function, is defined by

\[
\Gamma(z) = \int_{0}^{\infty} u^{z-1} e^{-u} du \quad (4.5.31)
\]
This definition implies that $\Gamma(n) = (n - 1)!$ when $n$ is a positive integer: $n = 1, 2, \ldots$. Indeed, we have by integration by parts

$$\Gamma(n + 1) = [-u^n e^{-u}]_0^\infty + n \int_0^\infty u^{n-1} e^{-u} \, du$$

$$= n\Gamma(n) \quad (4.5.32)$$

and when $z = 1$, Eq. (4.5.31) directly yields $\Gamma(1) = 1$. Equation (4.5.32) is a difference equation, which, when supplemented with the starting condition $\Gamma(1) = 1$, can be solved for all $n$. So when $n = 1$, Eq. (4.5.32) yields $\Gamma(2) = \Gamma(1) = 1$; when $n = 2$, $\Gamma(3) = 2\Gamma(2) = 2!$, . . . , and by induction, $\Gamma(n) = (n - 1)!$ for positive integer $n$. We often use Eq. (4.5.31) for general values of $z$, requiring only that $\text{Re} \, z > 0$ in order for there to be an integrable singularity at $u = 0$. With the definition (4.5.31) and rescaling $rx = u$ we have

$$f(x) = \left( \frac{\sin a\pi}{\pi} \right) x^{a-1} \Gamma(1 - a)$$

The Laplace transform of a derivative is readily calculated.

$$\hat{F}_1(s) = \int_0^\infty f'(x) e^{-sx} \, dx$$

$$= [f(x)e^{-sx}]_0^\infty + s \int_0^\infty f(x)e^{-sx} \, dx$$

$$= s\hat{F}(s) - f(0)$$

By integration by parts $n$ times it is found that

$$\hat{F}_n(s) = \int_0^\infty f^{(n)}(x)e^{-sx} \, dx$$

$$= s^n\hat{F}(s) - f^{(n-1)}(0) - sf^{(n-2)}(0) - \cdots - s^{n-2} f'(0) - s^{n-1} f(0) \quad (4.5.33)$$

The Laplace transform analog of the convolution result for Fourier transforms (4.5.17) takes the following form. Define

$$h(x) = \int_0^x g(x') f(x - x') \, dx' \quad (4.5.34)$$

We show that the Laplace transform of $h$ is the product of the Laplace transform
of $g$ and $f$.

$$\hat{H}(s) = \int_0^\infty h(x) e^{-sx} \, dx$$

$$= \int_0^\infty dx \, e^{-sx} \int_0^x g(x') f(x - x') \, dx'$$

By interchanging integrals, we find that

$$\hat{H}(s) = \int_0^\infty dx' \, g(x') \int_x^\infty dx \, e^{-sx} \, f(x - x')$$

$$= \int_0^\infty dx' \, g(x') e^{-sx'} \int_{x'}^{\infty} dx \, e^{-s(x-x')} \, f(x - x')$$

$$= \int_0^\infty dx' \, g(x') e^{-sx'} \int_0^{\infty} du \, e^{-su} \, f(u)$$

hence

$$\hat{H}(s) = \hat{G}(s) \hat{F}(s) \quad (4.5.35)$$

The convolution formulae (4.5.34–4.5.35) can be used in a variety of ways. We note that if $\hat{F}(s) = 1/s$, then Eq. (4.5.27) implies $f(x) = 1$. We use this in the following example.

**Example 4.5.4** Evaluate $h(x)$ where the Laplace transform of $h(x)$ is given by

$$\hat{H}(s) = \frac{1}{s(s^2 + 1)}$$

We have two functions:

$$\hat{F}(s) = \frac{1}{s}, \quad \hat{G}(s) = \frac{1}{s^2 + 1}$$

Using the result of Example 4.5.2 shows that $g(x) = \sin x$; hence Eq. (4.5.35) implies that

$$h(x) = \int_0^x \sin x' \, dx' = 1 - \cos x$$

This may also be found by the partial fractions decomposition

$$\hat{H}(s) = \frac{1}{s} - \frac{s}{s^2 + 1}$$

and noting that $\hat{F}(s) = \frac{s}{s^2 + w^2}$ has, as its Laplace transform, $f(x) = \cos wx$. 


4.5 Fourier and Laplace Transforms

Problems for Section 4.5

1. Obtain the Fourier transforms of the following functions:

(a) \( e^{-|x|} \)  
(b) \( \frac{1}{x^2 + a^2}, \quad a^2 > 0 \)

(c) \( \frac{1}{(x^2 + a^2)^2}, \quad a^2 > 0 \)  
(d) \( \frac{\sin(ax)}{(x + b)^2 + c^2}, \quad a, b, c > 0 \).

2. Obtain the inverse Fourier transform of the following functions:

(a) \( \frac{1}{k^2 + w^2}, \quad w^2 > 0 \)  
(b) \( \frac{1}{(k^2 + w^2)^2}, \quad w^2 > 0 \)

Note the duality between direct and inverse Fourier transforms.

3. Show that the Fourier transform of the “Gaussian”

\[ f(x) = \exp(-\frac{(x-x_0)^2}{a^2}) \quad x_0, \ a \ \text{real} \]

is also a Gaussian:

\[ \hat{F}(k) = a\sqrt{\pi}e^{-\frac{(ka/2)^2}{2}}e^{-ikx_0} \]

4. Obtain the Fourier transform of the following functions, and thereby show that the Fourier transforms of hyperbolic secant functions are also related to hyperbolic secant functions.

(a) \( \text{sech}[a(x-x_0)]e^{i\omega x}, \quad a, x_0, \omega \ \text{real} \)

(b) \( \text{sech}^2[a(x-x_0)], \quad a, x_0 \ \text{real} \)

5. (a) Obtain the Fourier transform of \( f(x) = (\sin \omega x)/(x) \), \( \omega > 0 \).

(b) Show that \( f(x) \) in part (a) is not \( L_1 \), that is, \( \int_{-\infty}^{\infty} |f(x)|dx \) does not exist. Despite this fact, we can obtain the Fourier transform, so \( f(x) \in L_1 \) is a sufficient condition, but is not necessary, for the Fourier transform to exist.

6. Suppose we are given the differential equation

\[ \frac{d^2u}{dx^2} - \omega^2 u = -f(x) \]

with \( u(x = \pm\infty) = 0, \omega > 0 \).

(a) Take the Fourier transform of this equation to find (using Eq. (4.5.16))

\[ \hat{U}(k) = \hat{F}(k)/(k^2 + \omega^2) \]

where \( \hat{U}(k) \) and \( \hat{F}(k) \) are the Fourier transform of \( u(x) \) and \( f(x) \), respectively.
(b) Use the convolution product (4.5.17) to deduce that

\[ u(x) = \frac{1}{2\omega} \int_{-\infty}^{\infty} e^{-\omega|x-\zeta|} f(\zeta) \, d\zeta \]

and thereby obtain the solution of the differential equation.

7. Obtain the Fourier sine transform of the following functions:

(a) \( e^{\omega x} \), \( \omega > 0 \)  
(b) \( \frac{x}{x^2 + 1} \)  
(c) \( \frac{\sin \omega x}{x^2 + 1} \), \( \omega > 0 \)

8. Obtain the Fourier cosine transform of the following functions:

(a) \( e^{-\omega x} \), \( \omega > 0 \)  
(b) \( \frac{1}{x^2 + 1} \)  
(c) \( \frac{\cos \omega x}{x^2 + 1} \), \( \omega > 0 \)

9. (a) Assume that \( u(\infty) = 0 \) to establish that

\[ \int_0^\infty \frac{d^2 u}{dx^2} \sin kx \, dx = k u(0) - k^2 \hat{U}_s(k) \]

where \( \hat{U}_s(k) = \int_0^\infty u(x) \sin kx \, dx \) is the sine transform of \( u(x) \) (Eq. (4.5.23)).

(b) Use this result to show that by taking the Fourier sine transform of

\[ \frac{d^2 u}{dx^2} - \omega^2 u = -f(x) \]

with \( u(0) = u_0 \), \( u(\infty) = 0 \), \( \omega > 0 \), yields, for the Fourier sine transform of \( u(x) \)

\[ \hat{U}_s(k) = \frac{u_0 k + \hat{F}_s(k)}{k^2 + \omega^2} \]

where \( \hat{F}_s(k) \) is the Fourier sine transform of \( f(x) \).

(c) Use the analog of the convolution product for the Fourier sine transform

\[ \frac{1}{2} \int_0^\infty [f(|x - \zeta|) - f(x + \zeta)] g(\zeta) \, d\zeta = \frac{2}{\pi} \int_0^\infty \sin kx \hat{F}_c(k) \hat{G}_s(k) \, dk, \]
where $\hat{F}_c(k)$ is the Fourier cosine transform of $f(x)$ and $\hat{G}_s(k)$ is the Fourier sine transform of $g(x)$, to show that the solution of the differential equation is given by

$$u(x) = u_0 e^{-\omega x} + \frac{1}{2\omega} \int_0^\infty (e^{-\omega |x-\zeta|} - e^{-\omega (x+\zeta)}) f(\zeta) d\zeta$$

(The convolution product for the sine transform can be deduced from the usual convolution product (4.5.17) by assuming in the latter formula that $f(x)$ is even and $g(x)$ is an odd function of $x$.)

10. (a) Assume that $u(\infty) = 0$ to establish that

$$\int_0^\infty \frac{d^2 u}{dx^2} \cos kx \, dx = -\frac{du}{dx}(0) - k^2 \hat{U}_c(k)$$

where $\hat{U}_c(k) = \int_0^\infty u(x) \cos kx \, dx$ is the cosine transform of $u(x)$.

(b) Use this result to show that taking the Fourier cosine transform of

$$\frac{d^2 u}{dx^2} - \omega^2 u = -f(x), \quad \text{with} \quad \frac{du}{dx}(0) = u_0', \quad u(\infty) = 0, \quad \omega > 0$$

yields, for the Fourier cosine transform of $u(x)$

$$\hat{U}_c(k) = \hat{F}_c(k) - \frac{u_0'}{k^2 + \omega^2}$$

where $\hat{F}_c(k)$ is the Fourier cosine transform of $f(x)$.

(c) Use the analog of the convolution product of the Fourier cosine transform

$$\frac{1}{2} \int_0^\infty (f(|x-\zeta|) + f(x+\zeta))g(\zeta) \, d\zeta$$

$$= \frac{2}{\pi} \int_0^\infty \cos kx \hat{F}_c(k) \hat{G}_c(k) \, dk$$

to show that the solution of the differential equation is given by

$$u(x) = -\frac{u_0'}{\omega} e^{-\omega x} + \frac{1}{2\omega} \int_0^\infty (e^{-\omega |x-\zeta|} + e^{-\omega |x+\zeta|}) f(\zeta) \, d\zeta$$

(The convolution product for the cosine transform can be deduced from the usual convolution product (4.5.17) by assuming in the latter formula that $f(x)$ and $g(x)$ are even functions of $x$.)
11. Obtain the inverse Laplace transforms of the following functions, assuming \( \omega, \omega_1, \omega_2 > 0 \).

\[
\begin{align*}
\text{(a)} & \quad \frac{s}{s^2 + \omega^2} \\
\text{(b)} & \quad \frac{1}{(s + \omega)^2} \\
\text{(c)} & \quad \frac{1}{(s + \omega)^n} \\
\text{(d)} & \quad \frac{s}{(s + \omega)^n} \\
\text{(e)} & \quad \frac{1}{(s + \omega_1)(s + \omega_2)} \\
\text{(f)} & \quad \frac{1}{s^2(s^2 + \omega^2)} \\
\text{(g)} & \quad \frac{1}{(s + \omega_1)^2 + \omega_2^2} \\
\text{(h)} & \quad \frac{1}{(s^2 - \omega^2)^2}
\end{align*}
\]

12. Show explicitly that the Laplace transform of the second derivative of a function of \( x \) satisfies

\[
\int_0^\infty f''(x)e^{-sx} \, dx = s^2 \hat{F}(s) - sf(0) - f'(0)
\]

13. Establish the following relationships, where we use the notation \( \mathcal{L}(f(x)) \equiv \hat{F}(s) \):

(a) \( \mathcal{L}(e^{ax}f(x)) = \hat{F}(s - a) \quad a > 0 \)

(b) \( \mathcal{L}(f(x-a)H(x-a)) = e^{-as} \hat{F}(s) \quad a > 0 \),

where

\[ H(x) = \begin{cases} 
1 & x \geq 0 \\
0 & x < 0
\end{cases} \]

(c) Use the convolution product formula for Laplace transforms to show that the inverse Laplace transform of \( \hat{H}(s) = 1/(s^2(s^2 + \omega^2)) \), \( \omega > 0 \) satisfies

\[ h(x) = \frac{1}{\omega} \int_0^x x' \sin \omega(x - x') \, dx' = \frac{x}{\omega^2} - \frac{\sin \omega x}{\omega^3} \]

Verify this result by using partial fractions.

14. (a) Show that the inverse Laplace transform of \( \hat{F}(s) = e^{-ax^{1/2}}/s \), \( a > 0 \), is given by

\[
f(x) = 1 - \frac{1}{\pi} \int_0^\infty \frac{\sin(ar^{1/2})}{r} e^{-rx} \, dr
\]

Note that the integral converges at \( r = 0 \).
4.5 Fourier and Laplace Transforms

(b) Use the definition of the error function integral

\[ \text{erfx} = \frac{2}{\sqrt{\pi}} \int_0^x e^{-r^2} \, dr \]

to show that an alternative form for \( f(x) \) is

\[ f(x) = 1 - \text{erf}\left( \frac{a}{2\sqrt{x}} \right) \]

(c) Show that the inverse Laplace transform of \( \hat{F}(s) = e^{-as^{1/2}}/s^{1/2}, a > 0 \), is given by

\[ f(x) = \frac{1}{\pi} \int_0^\infty \frac{\cos(ar^{1/2})}{r^{1/2}} e^{-rx} \, dr \]

or the equivalent forms

\[ f(x) = \frac{2}{\pi \sqrt{x}} \int_0^\infty \cos au \sqrt{u} e^{-u^2} \, du = \frac{1}{\sqrt{\pi x}} e^{-a^2/4x} \]

Verify this result by taking the derivative with respect to \( a \) in the formula of part (a).

(d) Follow the procedure of part (c) and show that the inverse Laplace transform of \( \hat{F}(s) = e^{-as^{1/2}} \) is given by

\[ f(x) = \frac{a}{2\sqrt{\pi x^{3/2}}} e^{-a^2/4x} \]

15. Show that the inverse Laplace transform of the function

\[ \hat{F}(s) = \frac{1}{\sqrt{s^2 + \omega^2}} \]

\( \omega > 0 \), is given by

\[ f(x) = \frac{1}{\pi} \int_{-\omega}^{\omega} \frac{e^{ixr}}{\sqrt{\omega^2 - r^2}} \, dr = \frac{2}{\pi} \int_0^1 \frac{\cos(\omega x \rho)}{\sqrt{1 - \rho^2}} \, d\rho \]

(The latter integral is a representation of \( J_0(\omega x) \), the Bessel function of order zero.) Hint: Deform the contour around the branch points \( s = \pm i\omega \), then show that the large contour at infinity and small contours encircling
\( \pm i \omega \) are vanishingly small. It is convenient to use the polar representations
\( s + i \omega_1 = r_1 e^{i \theta_1} \) and
\( s - i \omega_2 = r_2 e^{i \theta_2} \), where
\(-3\pi/2 < \theta_i \leq \pi/2, i = 1, 2,\)
and
\( (s^2 + \omega^2)^{1/2} = \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2} \). The contributions on both sides of the cut add to give the result.

16. Show that the inverse Laplace transform of the function
\( \hat{F}(s) = (\log s) / (s^2 + \omega^2), \omega > 0 \) is given by
\[
\hat{F}(s) = \frac{\pi}{2\omega} \cos \omega x + (\log \omega) \frac{\sin \omega x}{\omega} - \int_0^\infty \frac{e^{-rx}}{r^2 + \omega^2} dr \quad \text{for } x > 0
\]
Hint: Choose the branch \( s = re^{i \theta}, -\pi \leq \theta < \pi \). Show that the contour at infinity and around the branch point \( s = 0 \) are vanishingly small. There are two contributions along the branch cut that add to give the second (integral) term; the first is due to the poles at \( s = \pm i \omega \).

17. (a) Show that the inverse Laplace transform of \( \hat{F}_1(s) = \log s \) is given by
\( f_1(x) = -1/x \).
(b) Do the same for \( \hat{F}_2(s) = \log(s + 1) \) to obtain \( f_2(x) = -e^{-x}/x \).
(c) Find the inverse Laplace transform \( \hat{F}(s) = \log((s + 1)/s) \) to obtain
\( f(x) = (1 - e^{-x})/x \), by subtracting the results of parts (a) and (b).
(d) Show that we can get this result directly, by encircling both the \( s = 0 \)
and \( s = -1 \) branch points and using the polar representations
\( s + 1 = r_1 e^{i \theta_1}, s = r_2 e^{i \theta_2}, -\pi \leq \theta_i < \pi, i = 1, 2.\)

18. Establish the following results by formally inverting the Laplace transform.
\[
\hat{F}(s) = \frac{1}{s} \frac{1 - e^{-\ell s}}{1 + e^{-\ell s}}, \quad \ell > 0,
\]
\[
f(x) = \sum_{n=1,3,5,...} \left( \frac{4}{n\pi} \right) \sin \frac{n\pi x}{\ell}
\]
Note that there are an infinite number of poles present in \( \hat{F}(s) \); consequently, a straightforward continuous limit as \( R \to \infty \) on a large semicircle will pass through one of these poles. Consider a large semicircle
\( C_{R_N}, \) where \( R_N \) encloses \( N \) poles (e.g. \( R_N = \frac{\pi i}{\ell} (N + \frac{1}{2}) \)) and show that as \( N \to \infty, R_N \to \infty, \) and the integral along \( C_{R_N} \) will vanish. Choosing appropriate sequences such as in this example, the inverse Laplace transform containing an infinite number of poles can be calculated.
19. Establish the following result by formally inverting the Laplace transform

\[ \hat{F}(s) = \frac{1}{s} \frac{\sinh(\sigma y)}{\sinh(s\ell)} \quad \ell > 0 \]

\[ f(x) = \frac{y}{\ell} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} \sin \left( \frac{n\pi y}{\ell} \right) \cos \left( \frac{n\pi x}{\ell} \right) \]

See the remark at the end of Problem 18, which explains how to show how the inverse Laplace transform can be proven to be valid in a situation such as this where there are an infinite number of poles.

*4.6 Applications of Transforms to Differential Equations*

A particularly valuable technique available to solve differential equations in infinite and semiinfinite domains is the use of Fourier and Laplace transforms. In this section we describe some typical examples. The discussion is not intended to be complete. The aim of this section is to elucidate the transform technique, not to detail theoretical aspects regarding differential equations. The reader only needs basic training in the calculus of several variables to be able to follow the analysis. We shall use various classical partial differential equations (PDEs) as vehicles to illustrate methodology. Herein we will consider well-posed problems that will yield unique solutions. More general PDEs and the notion of well-posedness are investigated in considerable detail in courses on PDEs.

**Example 4.6.1** Steady state heat flow in a semiinfinite domain obeys Laplace’s equation. Solve for the bounded solution of Laplace’s equation

\[ \frac{\partial^2 \phi(x, y)}{\partial x^2} + \frac{\partial^2 \phi(x, y)}{\partial y^2} = 0 \quad (4.6.1) \]

in the region \(-\infty < x < \infty, y > 0\), where on \(y = 0\) we are given \(\phi(x, 0) = h(x)\) with \(h(x) \in L^1 \cap L^2\) (i.e., \(\int_{-\infty}^{\infty} |h(x)| \, dx < \infty\) and \(\int_{-\infty}^{\infty} |h(x)|^2 \, dx < \infty\)).

This example will allow us to solve Laplace’s Equation (4.6.1) by Fourier transforms. Denoting the Fourier transform in \(x\) of \(\phi(x, y)\) as \(\hat{\phi}(k, y)\):

\[ \hat{\phi}(k, y) = \int_{-\infty}^{\infty} e^{-ikx} \phi(x, y) \, dx \]

taking the Fourier transform of Eq. (4.6.1), and using the result from Section 4.5 for the Fourier transform of derivatives, Eqs. (4.5.16a,b) (assuming the validity