Introduction to Game Theory
Lecture Note 1: Strategic-Form Games and Nash Equilibrium (1)

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• Game theory studies rational players’ behavior when they engage in strategic interactions.

• Rational choice: the action chosen by a decision maker is better or at least as good as every other available action, according to her preferences.

• Preferences are rational if they satisfy
  - **Completeness**: between any $x$ and $y$ in a set, $x \succ y$ ($x$ is preferred to $y$), $y \succ x$, or $x \sim y$ (indifferent)
  - **Transitivity**: $x \succeq y$ and $y \succeq z \Rightarrow x \succeq z$ ($\succeq$ means $\succ$ or $\sim$)

  ⇒ Say apple $\succ$ banana, and banana $\succ$ orange, then apple $\succ$ orange
Preferences and payoff functions (utility functions)

• No other restrictions on preferences. Preferences can be altruistic.
  ▶ But individual rationality does not necessarily mean collective rationality.

• Payoff function/utility function: \( u(x) \geq u(y) \) iff \( x \succeq y \)

• For now we only deal with ordinal (as opposed to cardinal) preferences, so you can use many different utility functions to represent the same preference relation.

  • Any strictly increasing transformation of the same utility function will do.
  • Say \( x \succ y \succ z \). Then \( u(x) = 3, u(y) = 2, u(z) = 1 \) represents the same preferences as \( u(x) = 100, u(y) = 10, u(z) = 2 \).
Types of games

- Games with complete information
  - Static games
  - Dynamic games

- Games with incomplete information
  - Static games (Bayesian games)
  - Dynamic games (dynamic Bayesian games)
Static games of complete information

- Static games: simultaneous-move, single-shot games
- Complete information: a player knows other players’ utility functions (and other characteristics that affect their decision making)
- We use the strategic form/normal form to represent a static game of complete information.
- Definition: A strategic-form game consists of
  1. a set of players
  2. for each player, a set of actions (i.e., strategies)
  3. for each player, preferences over the set of action/strategy profiles
Static games of complete information

• **Strategy profile**: a list of all the player’s strategies
  ▶ E.g, my strategies: left or right; your strategies: up or down
  ▶ Strategy/action profiles: (left, up), (left, down), any other?

• Preferences are over strategy profiles rather than one’s own actionsstrategies.

• In single-shot games, actions are equivalent to strategies.
Illustration: prisoner's dilemma

- Players: two suspects, 1 and 2
- Actions: \{stay silent, confess\}
- Preferences:
  - $u_1(\text{confess, silent}) > u_1(\text{silent, silent}) > u_1(\text{confess, confess}) > u_1(\text{silent, confess})$
  - $u_2(\text{silent, confess}) > u_2(\text{silent, silent}) > u_2(\text{confess, confess}) > u_2(\text{confess, silent})$
- Game representation

<table>
<thead>
<tr>
<th>Suspect 1</th>
<th>silent</th>
<th>confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>silent</td>
<td>0, 0</td>
<td>−2, 1</td>
</tr>
<tr>
<td>confess</td>
<td>1, −2</td>
<td>−1, −1</td>
</tr>
</tbody>
</table>
Nash equilibrium

- Definition: A strategy profile $a^*$ is a **Nash equilibrium** if, for every player $i$ and every strategy $a_i$ of player $i$, $a^*$ is at least as good for player $i$ as the strategy profile $(a_i, a^*_i)$ in which player $i$ chooses $a_i$ while every other player $j$ chooses $a_j^*$.

- In other words: $u_i(a^*) \geq u_i(a_i, a^*_j)$ for every strategy $a_i$ of every player $i$.

- In plain English: no one can do better by unilaterally deviating from the strategy profile.

- A Nash equilibrium is a **steady state**. It embodies a stable “social norm”: if everyone else sticks to it, no one has incentive to deviate from it.
What’s the Nash equilibrium in PD?

<table>
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<th>confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>silent</td>
<td>0, 0</td>
<td>−2, 1</td>
</tr>
<tr>
<td>confess</td>
<td>1, −2</td>
<td>−1, −1</td>
</tr>
</tbody>
</table>

Only the strategy profile (confess, confess) is a NE.

In PD each player has an **dominant strategy**: a strategy that is better for a player regardless of what other players do.
Prisoner’s dilemma cont.

- Tragedy of the PD game: there is an outcome that is better for BOTH players, but they just cannot achieve it.
- Would communication between the two players help them?
  - Watch a real game: http://www.youtube.com/watch?v=p3Uos2fzIJO&feature=player_embedded
- Applications: tragedy of commons; arms race
Battle of sexes

- He wants to watch soccer, she wants to watch ballet, but they would rather be together than separate.

<table>
<thead>
<tr>
<th></th>
<th>soccer</th>
<th>ballet</th>
</tr>
</thead>
<tbody>
<tr>
<td>soccer</td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>ballet</td>
<td>0, 0</td>
<td>1, 2</td>
</tr>
</tbody>
</table>

- What are the Nash equilibria?
- 2 Nash equilibria: (soccer, soccer); (ballet, ballet)
- BoS models situations in which two parties want to cooperate but disagree on which point to cooperate.
Matching pennies

- A purely conflictual game (PD and BoS have elements of cooperation)

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<thead>
<tr>
<th></th>
<th>Head</th>
<th>Tail</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head</td>
<td>1, -1</td>
<td>-1, 1</td>
</tr>
<tr>
<td>Tail</td>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
</tbody>
</table>

- Player 1 wants to take the same action as player 2, but player 2 wants to take the opposite action.
- Any (pure-strategy) Nash equilibrium? 
  \[\Rightarrow\text{ No.}\]
Two hunters can succeed in catching a stag if they all exert efforts, but each can catch a hare alone.

<table>
<thead>
<tr>
<th></th>
<th>Hunter 1</th>
<th>Hunter 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>stag</td>
<td>2, 2</td>
<td>0, 1</td>
</tr>
<tr>
<td>hare</td>
<td>1, 0</td>
<td>1, 1</td>
</tr>
</tbody>
</table>

What are the Nash equilibria?
⇒ (stag, stag) and (hare, hare)

Application: cooperative project; security dilemma
The chicken game (hawk-dove)

- Two drivers drive towards each other on a single lane. If neither swerves, they collide and may die; if one swerves while the other does not, the one who swerves loses face while the other gains respect.

<table>
<thead>
<tr>
<th>Driver 1</th>
<th>Driver 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>straight</td>
<td>-10, -10</td>
</tr>
<tr>
<td>swerve</td>
<td>-1, 1</td>
</tr>
</tbody>
</table>

- Application: brinkmanship
- Reducing options in a chicken game: throwing away the steering wheel? Burning the bridge after crossing the river?
Coordination and the focal point

- A coordination game: choosing a restaurant
  
<table>
<thead>
<tr>
<th></th>
<th>Italian</th>
<th>Japanese</th>
</tr>
</thead>
<tbody>
<tr>
<td>Italian</td>
<td>1, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>Japanese</td>
<td>0, 0</td>
<td>1, 1</td>
</tr>
</tbody>
</table>

- NE: (Italian, Italian); (Japanese, Japanese)

- **Focal point**: in some real-life situations players may be able to coordinate on a particular equilibrium in a multiple equilibria game, by using information that is abstracted away from the strategic form.

  ▶ Schelling’s experiment about meeting in New York
Osborne (2004) exercise 33.1: Each of $n$ people chooses whether to contribute a fixed amount toward the provision of a public good. The good is provided iff at least $k$ people contribute, where $2 \leq k \leq n$; if it is not provided, contribution are not refunded. Each person ranks outcomes from best to worst as follows: (a) any outcome in which the good is provided and she does not contribute; (b) any outcome in which the good is provided and she contributes; (c) any outcome in which the good is not provided and she does not contribute; (d) any outcome in which the good is not provided and she contributes. Formulate this situation as a strategic game and find the NE.
Public good provision: strategic form

- Players: the $n$ people
- Actions: each player’s set of action is \{contribute, not contribute\}
- Preferences: $U_i(a) > U_i(b) > U_i(c) > U_i(d)$
• Is there a NE in which more than $k$ people contribute? One in which $k$ people contribute? One in which fewer than $k$ contribute?

• NE: $k$ people contribute; none contributes
Strict and non-strict equilibria

- If an action profile \( a^* \) is a NE, then \( u_i(a^*) \geq u_i(a_i, a^*_{-i}) \) for every action \( a_i \) of every player \( i \).
- An equilibrium is strict if each player’s equilibrium action is better than all her other actions. Or, \( u_i(a^*) > u_i(a_i, a^*_{-i}) \) for every action \( a_i \neq a_i^* \) of player \( i \).
- A variant of the prisoner’s dilemma game

<table>
<thead>
<tr>
<th>Player 1</th>
<th>split</th>
<th>steal</th>
</tr>
</thead>
<tbody>
<tr>
<td>split</td>
<td>5, 5</td>
<td>0, 10</td>
</tr>
<tr>
<td>steal</td>
<td>10, 0</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

- How many Nash equilibria? Any strict NE?
  \[ \Rightarrow 3 \text{ and } 0. \]