CSE 135: Introduction to Theory of Computation
(A taste of) Chomsky Hierarchy

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03-19-2015

Grammars

Definition

A grammar is $G = (V, \Sigma, R, S)$, where

- V is a finite set of variables/non-terminals
- Σ is a finite set of terminals
- $S \in V$ is the start symbol
- $R \subseteq (\Sigma \cup V)^* \times (\Sigma \cup V)^*$ is a finite set of rules/productions

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We say $\gamma_1 \alpha \gamma_2 \Rightarrow_G \gamma_1 \beta \gamma_2$ iff $(\alpha \to \beta) \in R$. And
$L(G) = \{w \in \Sigma^* \mid S \overset{*}{\Rightarrow}_G w\}$

Example

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Consider the grammar G with $\Sigma = \{a\}$ with

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\begin{array}{lll}
S \rightarrow \$Ca\# \mid a \mid \epsilon & Ca \rightarrow aaC & \$D \rightarrow \$C \\
C\# \rightarrow D\# \mid E & aD \rightarrow Da & aE \rightarrow Ea \\
\$E \rightarrow \epsilon & &
\end{array}
$$

The following are derivations in this grammar

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S \Rightarrow \$Ca\# \Rightarrow \$aaC\# \Rightarrow \$aaE \Rightarrow \$aEa \Rightarrow \$Eaa \Rightarrow aa
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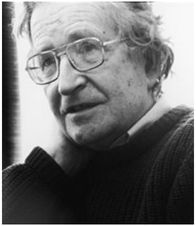
$L(G) = \{a^i \mid i \text{ is a power of } 2\}$

Grammars for each task

- What is the expressive power of these grammars?

Grammars for each task



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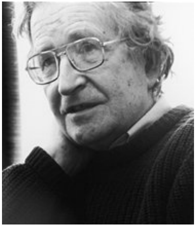

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- Restricting the types of rules, allows one to describe different aspects of natural languages
- These grammars form a hierarchy

Type 0 Grammars

Definition

Type 0 grammars are those where the rules are of the form

$$\alpha \to \beta$$

where $\alpha, \beta \in (\Sigma \cup V)^*$

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Expressive Power of Type 0 Grammars

Theorem

L is recursively enumerable iff there is a type 0 grammar G such that $L = L(G)$.

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Thus, type 0 grammars are as powerful as Turing machines.

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Example

Consider the grammar G with $\Sigma = \{a, b, c\}$, $V = \{S, B, C, H\}$ and

$$\begin{array}{lll} S \to aSBC \mid aBC & CB \to HB & HB \to HC \\ HC \to BC & aB \to ab & bB \to bb \\ bC \to bc & cC \to cc & \end{array}$$

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$L(G) = \{a^n b^n c^n \mid n \geq 0\}$

Context Sensitivity

Normal Form for Type 1 grammars

For every Type 1 grammar G, there is a grammar (in normal form) G' such that $L(G) = L(G')$ and all the rules of G' are of the form

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So, rules of G' replace a variable A by β in the context $\alpha_1 \Box \alpha_2$.
Thus, the class of language described by Type 1 grammars are called context-sensitive languages.

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$L(G) = \{w \in \{0, 1\}^* \mid w \text{ has an odd number of 0s}\}$

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Let $G = (V, \Sigma, R, S)$ be a type 3 grammar. Consider the NFA $M = (Q, \Sigma, \delta, q_0, F)$ where

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- $Q = V \cup \{q_F\}$, where $q_F \notin V$
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- $\delta(A, a) = \{B \mid \text{if } A \to aB \in R\} \cup \{q_F \mid \text{if } A \to a \in R\}$ for $A \in V$. And $\delta(q_F, a) = \emptyset$ for all a.

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$L(M) = L(G)$ as $\forall A \in V$, $\forall w \in \Sigma^*$, $A \overset{*}{\Rightarrow}_G w$ iff $q_F \in \hat{\Delta}(A, w)$. $\cdot\cdot\!\!\longrightarrow$

Type 3 Grammars and Regularity

NFA to Grammars

Proof (contd).

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a NFA recognizing L. Consider $G = (V, \Sigma, R, S)$ where

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- $V = Q$

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- $V = Q$
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- $q_1 \rightarrow aq_2 \in R$ iff $q_2 \in \delta(q_1, a)$ and $q \rightarrow \epsilon \in R$ iff $q \in F$.

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- $q_1 \to aq_2 \in R$ iff $q_2 \in \delta(q_1, a)$ and $q \to \epsilon \in R$ iff $q \in F$.

We can show, for any $q, q' \in Q$ and $w \in \Sigma^*$, $q' \in \hat{\Delta}(q, w)$ iff $q \overset{*}{\Rightarrow}_G wq'$. Thus, $L(M) = L(G)$. $\square$

Grammars and their Languages

Grammar	Rules	Languages
Type 3	$A \rightarrow aB$ or $A \rightarrow a$	Regular
Type 2	$A \rightarrow \alpha$	Context Free
Type 1	$\alpha \rightarrow \beta$ with $\lvert\alpha\rvert \leq \lvert\beta\rvert$	Context Sensitive
Type 0	$\alpha \rightarrow \beta$	Recursively Enumerable

In the above table, $\alpha, \beta \in (\Sigma \cup V)^*$, $A, B \in V$ and $a \in \Sigma \cup \{\epsilon\}$.

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Overview of Languages