CSE 135: Introduction to Theory of Computation
Closure Properties of CFLs

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Union of CFLs

Let L_1 be language recognized by $G_1 = (V_1, \Sigma_1, R_1, S_1)$ and L_2 the language recognized by $G_2 = (V_2, \Sigma_2, R_2, S_2)$

Is $L_1 \cup L_2$ a context free language?

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Closure of CFLs under Union

$G = (V, \Sigma, R, S)$ such that $L(G) = L(G_1) \cup L(G_2)$:

- $V = V_1 \cup V_2 \cup \{S\}$ (the three sets are disjoint)
- $\Sigma = \Sigma_1 \cup \Sigma_2$
- $R = R_1 \cup R_2 \cup \{S \rightarrow S_1|S_2\}$

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- But $L_1 \cap L_2 = \{a^n b^n c^n \mid n \geq 0\}$ is not a CFL. □

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Why does this construction not work for intersection of two CFLs?

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But $\overline{L} = \{ww \mid w \in \{a, b\}^*\}$ is not a CFL! (Why?) □

Set Difference

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$L \setminus R = L \cap \overline{R}$ □

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Solution: Check if the start symbol S is generating. How long does that take?

Determining generating symbols

Algorithm

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Gen = {}
for every rule A → x where x ∈ Σ*
    Gen = Gen ∪ {A}
repeat
    for every rule A → γ
        if all variables in γ are generating then
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until Gen does not change
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- Each iteration of repeat-until loop discovers a new variable. So number of iterations is $O(n)$. And total is $O(n^2)$.

Membership Problem

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Central question in parsing.

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- We will see an algorithm that runs in $O(n^3)$ time (the constant will depend on k).

First Ideas

Notation

Suppose $w = w_1 w_2 \cdots w_n$, where $w_i \in \Sigma$. Let $w_{i,j}$ denote the substring of w starting at position i of length j. Thus,

$w_{i,j} = w_i w_{i+1} \cdots w_{i+j-1}$

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How do we determine if $A \stackrel{*}{\Rightarrow} w_{i,j}$ for every A, i, j?

Base Case

Substrings of length 1

Observation

For any A, i, $A \stackrel{*}{\Rightarrow} w_{i,1}$ iff $A \rightarrow w_{i,1}$ is a rule.

Base Case

Substrings of length 1

Observation

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- Since G is in Chomsky Normal Form, G does not have any ϵ-rules, nor any unit rules.

Thus, for each A and i, one can determine if $A \stackrel{*}{\Rightarrow} w_{i,1}$.

Inductive Step

Longer substrings

Suppose for every variable X and every $w_{i,\ell}$ ($\ell < j$) we have determined if $X \overset{*}{\Rightarrow} w_{i,\ell}$

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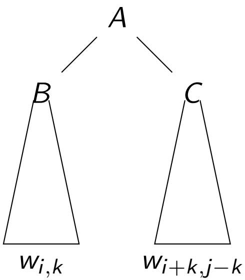

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- $A \stackrel{*}{\Rightarrow} w_{i,j}$ iff there are variables B and C and some $k < j$ such that $A \to BC$ is a rule, and $B \stackrel{*}{\Rightarrow} w_{i,k}$ and $C \stackrel{*}{\Rightarrow} w_{i+k,j-k}$

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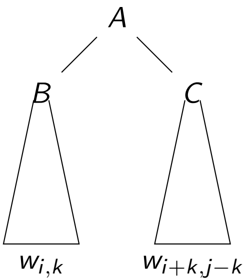

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- Since k and $j - k$ are both less than j, we can inductively determine if $A \overset{*}{\Rightarrow} w_{i,j}$.

Cocke-Younger-Kasami (CYK) Algorithm

Algorithm maintains $X_{i,j} = \{A \mid A \stackrel{*}{\Rightarrow} w_{i,j}\}$.

```
Initialize:   X_{i,1} = {A | A → w_{i,1}}
for j = 2 to n do
    for i = 1 to n − j + 1 do
        X_{i,j} = ∅
        for k = 1 to j − 1 do
            X_{i,j} = X_{i,j} ∪ {A | A → BC, B ∈ X_{i,k}, C ∈ X_{i+k,j−k}}
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    for k `= 1 to` $j - 1$ **do**
      $X_{i,j} = X_{i,j} \cup \{A \mid A \rightarrow BC,\ B \in X_{i,k},\ C \in X_{i+k,j-k}\}$

Correctness: After each iteration of the outermost loop, $X_{i,j}$ contains exactly the set of variables A that can derive $w_{i,j}$, for each i.

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Consider grammar
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j/i	1	2	3	4	5
5					
4					
3					
2					
1	$\{B\}$	$\{A, C\}$	$\{A, C\}$	$\{B\}$	$\{A, C\}$
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Given a CFGs G_1 and G_2

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All these problems are undecidable.