CSE 135: Introduction to Theory of Computation
Pushdown Automata

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  - can read/erase only the top of the stack: pop
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- On longer inputs, automaton may have more items in the stack

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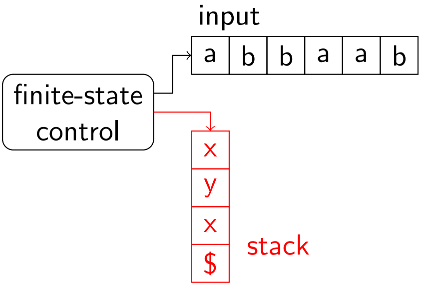
Pushdown Automata (PDA)



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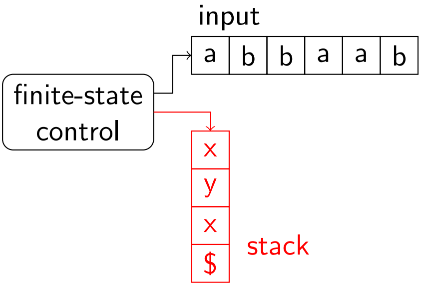

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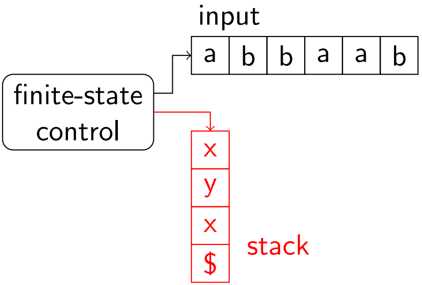

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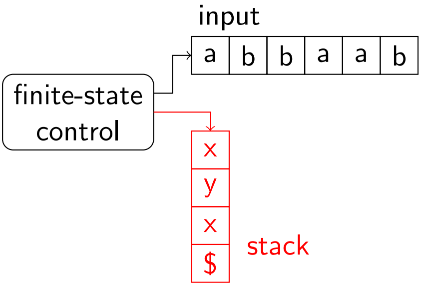

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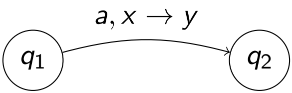

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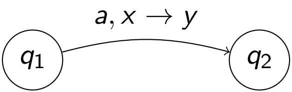


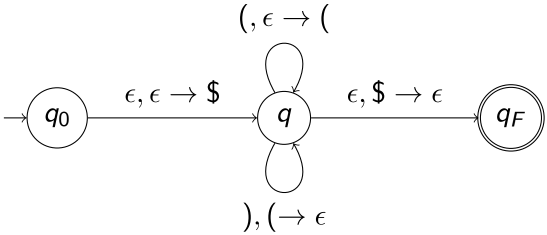
If at q_1, with next input symbol a and top of stack x, then can consume a, pop x, push y onto stack and move to q_2 (any of a, x, y may be ϵ)

Pushdown Automata (PDA): Formal Definition

A PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$ where

- Q = Finite set of states
- Σ = Finite input alphabet
- Γ = Finite stack alphabet
- q_0 = Start state
- $F \subseteq Q$ = Accepting/final states
- $\delta : Q \times (\Sigma \cup \{\epsilon\}) \times (\Gamma \cup \{\epsilon\}) \to \mathcal{P}(Q \times (\Gamma \cup \{\epsilon\}))$

Matching Parenthesis: PDA construction



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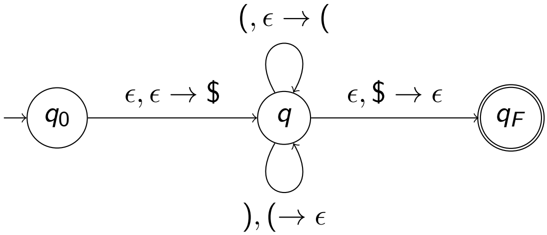

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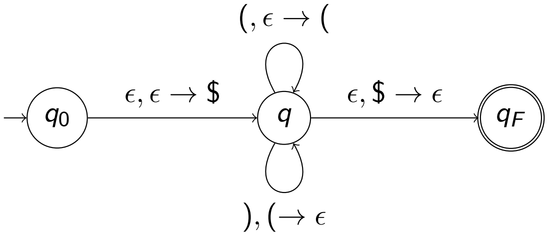

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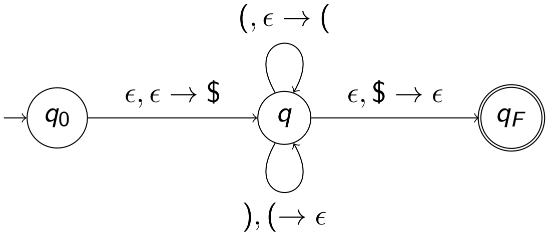

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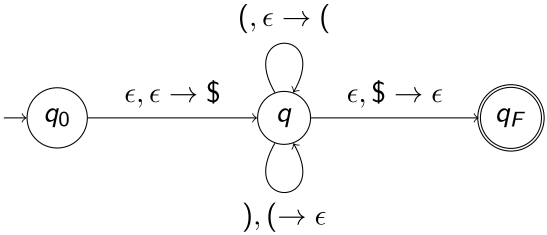

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- Pop \$ and move to final state q_F

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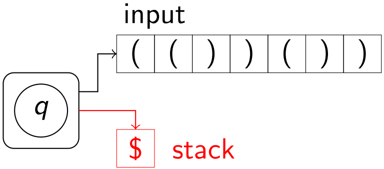

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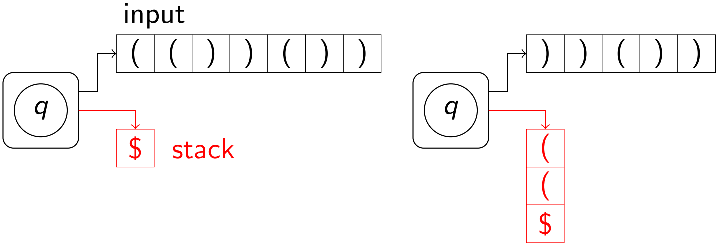

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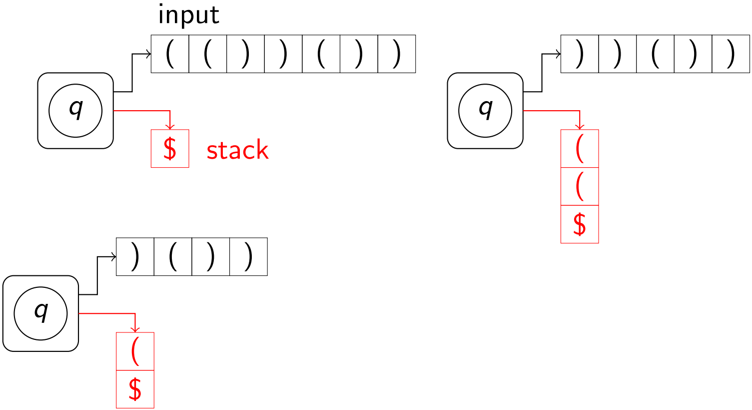

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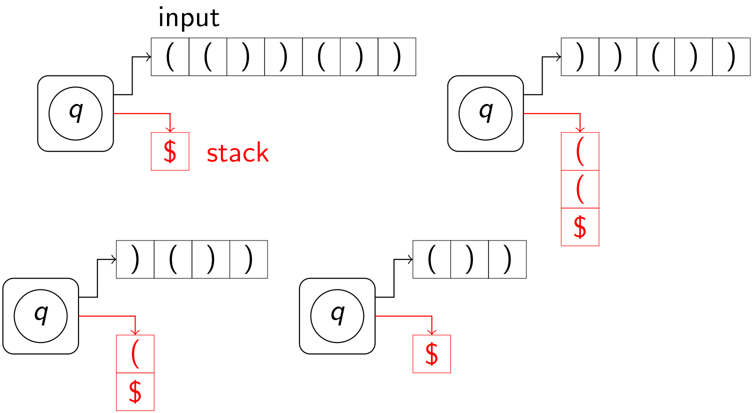

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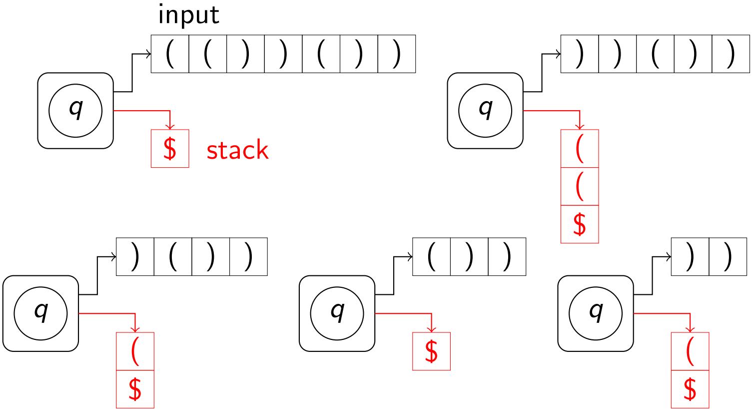

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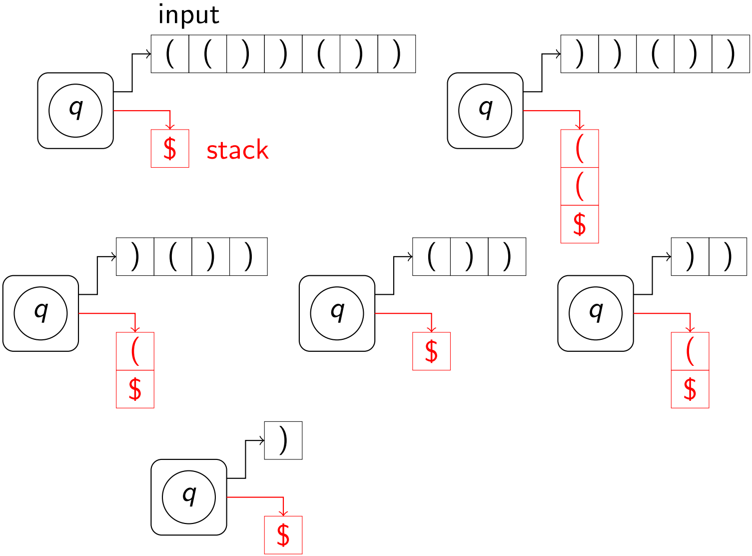

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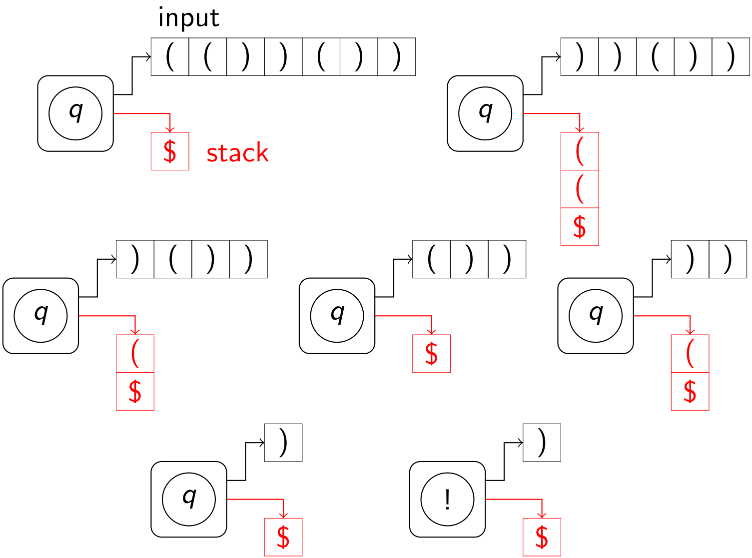

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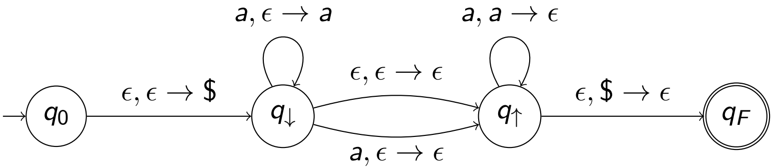

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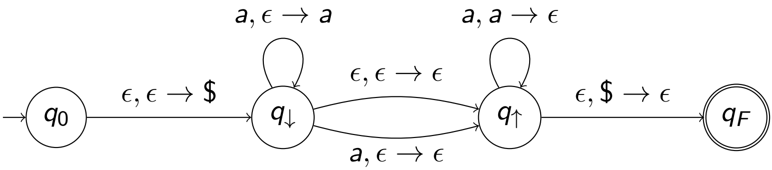

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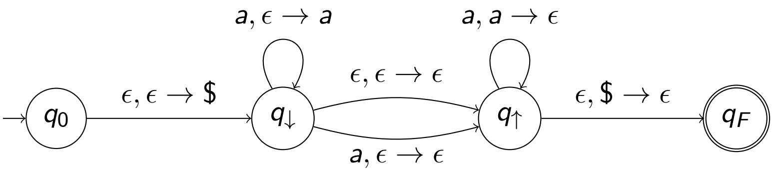

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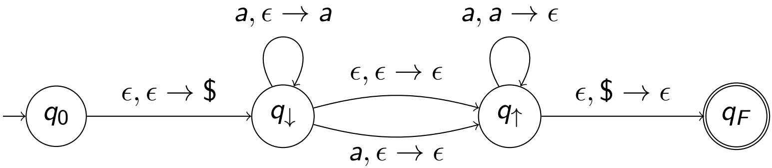

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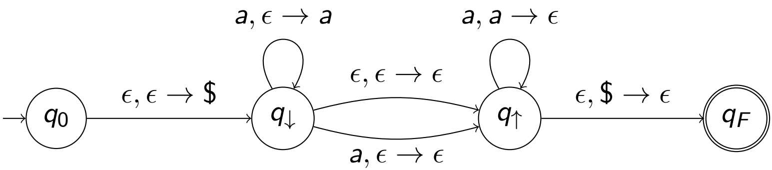

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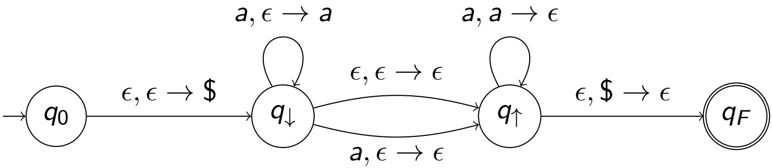

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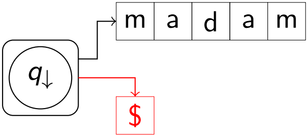

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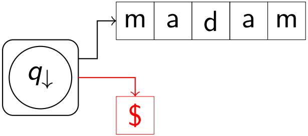

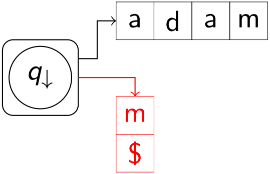

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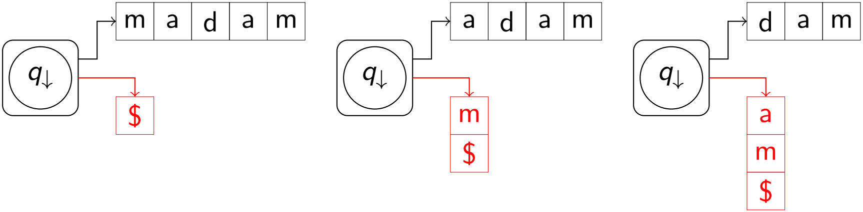

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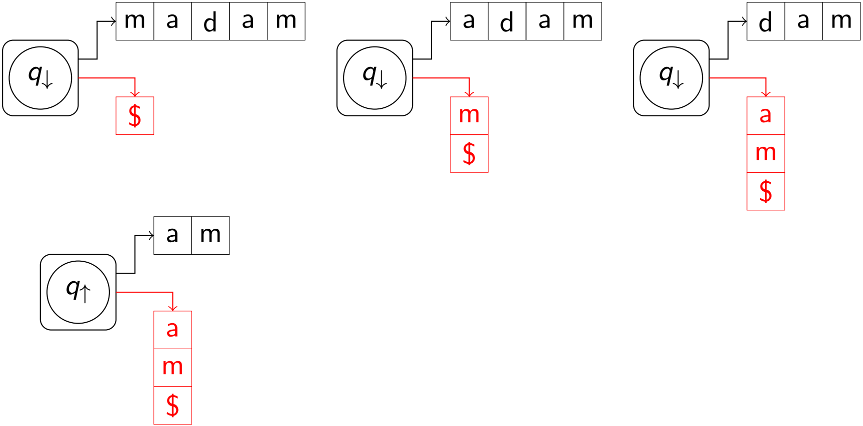

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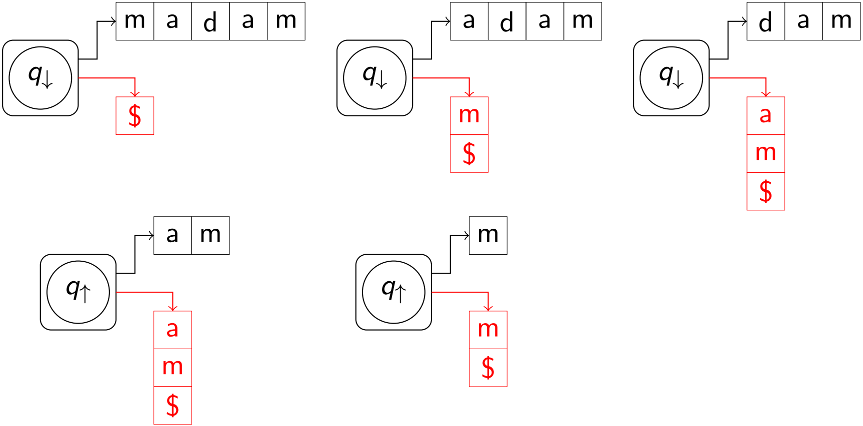

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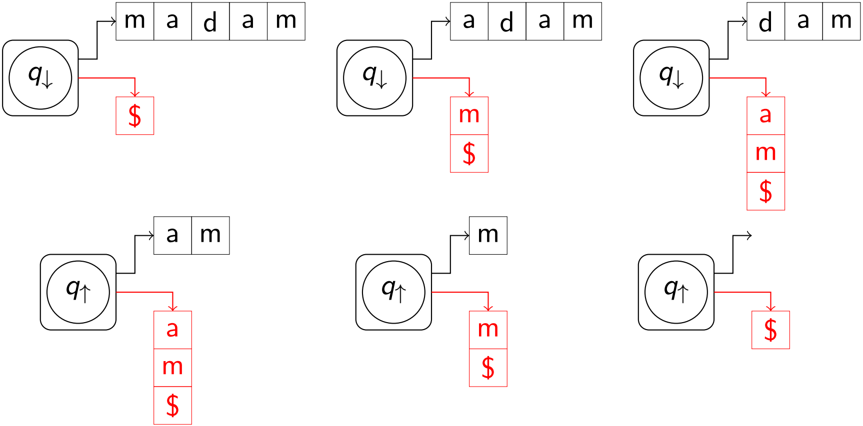

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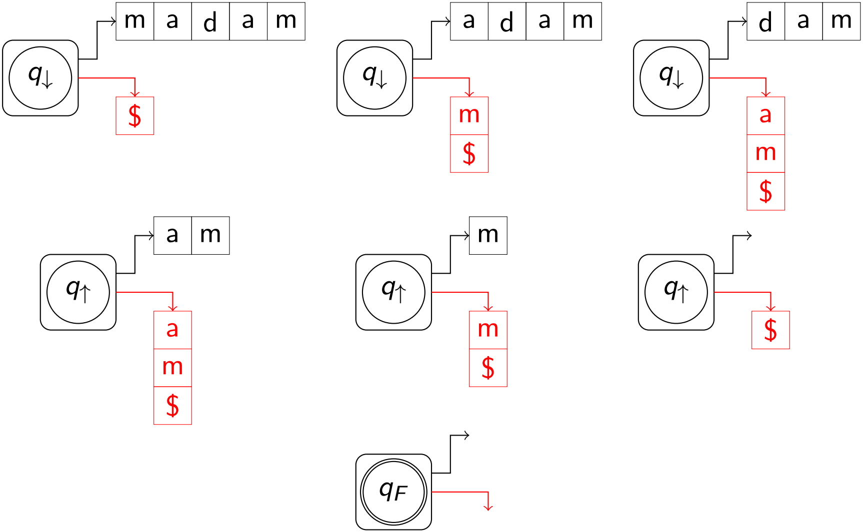

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Definition

An instantaneous description of a PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$ is a pair $\langle q, \sigma \rangle$, where $q \in Q$ and $\sigma \in \Gamma^*$

Computation

Definition

For a PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$, string $w \in \Sigma^*$, and instantaneous descriptions $\langle q_1, \sigma_1 \rangle$ and $\langle q_2, \sigma_2 \rangle$, we say $\langle q_1, \sigma_1 \rangle \xrightarrow{w}_P \langle q_2, \sigma_2 \rangle$ iff there is a sequence of instanteous descriptions $\langle r_0, s_0 \rangle, \langle r_1, s_1 \rangle, \ldots \langle r_k, s_k \rangle$ and a sequence $x_1, x_2, \ldots x_k$, where for each i, $x_i \in \Sigma \cup \{\epsilon\}$, such that

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- $r_0 = q_1$, and $s_0 = \sigma_1$,
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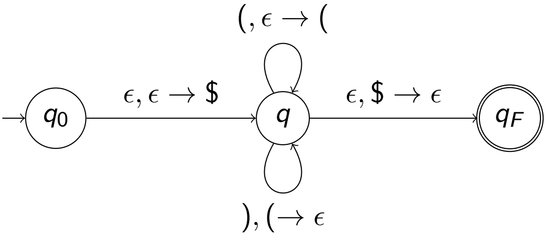
- for every i, $(r_{i+1}, b) \in \delta(r_i, x_{i+1}, a)$ such that $s_i = as$ and $s_{i+1} = bs$, where $a, b \in \Gamma \cup \{\epsilon\}$ and $s \in \Gamma^*$

Example of Computation

Example



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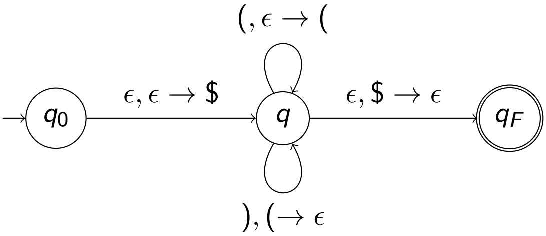

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$$\langle q_0, \epsilon \rangle \xrightarrow{x_1=\epsilon} \langle q, \$ \rangle \xrightarrow{x_2=(} \langle q, (\$ \rangle \xrightarrow{x_3=(} \langle q, ((\$ \rangle \xrightarrow{x_4=)} \langle q, (\$ \rangle \xrightarrow{x_5=(} \langle q, ((\$ \rangle$$

Acceptance/Recognition

Definition

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The language recognized/accepted by a PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$ is $L(P) = \{w \in \Sigma^* \mid P \text{ accepts } w\}$.

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Definition

The language recognized/accepted by a PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$ is $L(P) = \{w \in \Sigma^* \mid P \text{ accepts } w\}$. A language L is said to be accepted/recognized by P if $L = L(P)$.

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Proof.

Skipped. $\square$