CSE 135: Introduction to Theory of Computation
Closure Properties of Regular Languages

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- Today: A variety of operations which preserve regularity
  - i.e., the universe of regular languages is closed under these operations

Closure Properties

Definition

Regular Languages are closed under an operation op on languages if

$$L_1, L_2, \ldots L_n \text{ regular} \implies L = \mathrm{op}(L_1, L_2, \ldots L_n) \text{ is regular}$$

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Regular languages are closed under

- "halving", i.e., L regular $\implies \frac{1}{2}L$ regular.
- "reversing", i.e., L regular $\implies L^{\text{rev}}$ regular.

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(Summarizing previous arguments.)

- L_1, L_2 regular $\implies$ $\exists$ regexes R_1, R_2 s.t. $L_1 = L(R_1)$ and $L_2 = L(R_2)$.
  - $\implies$ $L_1 \cup L_2 = L(R_1 \cup R_2)$ $\implies$ $L_1 \cup L_2$ regular.

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  - $\implies L_1 \cup L_2 = L(R_1 \cup R_2) \implies L_1 \cup L_2$ regular.
  - $\implies L_1 \circ L_2 = L(R_1 \circ R_2) \implies L_1 \circ L_2$ regular.
  - $\implies L_1^* = L(R_1^*) \implies L_1^*$ regular. □

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What happens if M (above) was an NFA?

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Is there a direct proof for intersection (yielding a smaller DFA)?

Cross-Product Construction

Let $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ be DFAs recognizing L_1 and L_2, respectively.

Idea: Run M_1 and M_2 in parallel on the same input and accept if both M_1 and M_2 accept.

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Consider $M = (Q, \Sigma, \delta, q_0, F)$ defined as follows

- $Q = Q_1 \times Q_2$
- $q_0 = \langle q_1, q_2 \rangle$
- $\delta(\langle p_1, p_2 \rangle, a) = \langle \delta_1(p_1, a), \delta_2(p_2, a) \rangle$
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M accepts $L_1 \cap L_2$ (exercise)

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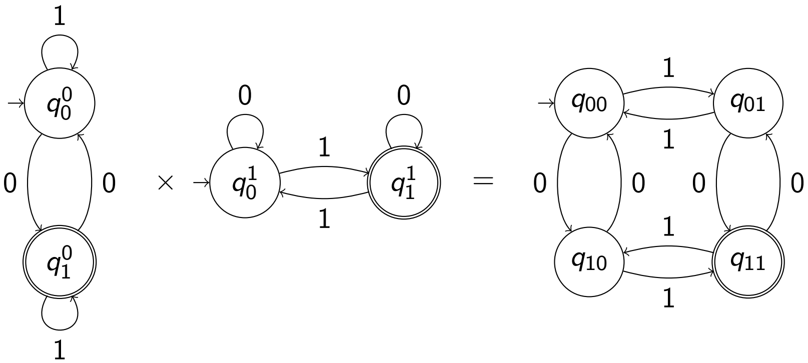
M accepts $L_1 \cap L_2$ (exercise)
What happens if M_1 and M_2 where NFAs? Still works! Set $\delta(\langle p_1, p_2 \rangle, a) = \delta_1(p_1, a) \times \delta_2(p_2, a)$.

An Example



Example (1)

$\Sigma = \{0, 1\}$. Say $w \in L$ iff $|w|$ is divisible by 7, but not 3, and w contains 1100 as a substring. Is L regular or not?
Answer: L is regular. Define

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$L = (L_1 \cap \overline{L_2}) \cap L_3$. So L is regular.

Example (2)

For any two languages L_1 and L_2, if $L_1 \cup L_2$ and L_1 are both regular, then L_2 must be regular. Is this claim true?
Answer: This is not true. Consider the case $L_1 = \Sigma^*$. Clearly, $L_1 \cup L_2 = L_1$ is regular. But we can pick any non-regular language L_2.

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