1. Recall that $\hat{\delta}(q, uv) = \hat{\delta}(\hat{\delta}(q, u), v)$ for all states q and strings u and v. Let's prove this in words. We know that in a DFA, the next state is determined by the current state and the input symbol the DFA reads. Hence, $\hat{\delta}(q, uv)$ denotes the DFA's state after reading $u_1, u_2, .., u_k$ and $v_1, v_2, ..., v_\ell$ when starting from q, where $u = u_1 u_2 ... u_k$ and $v = v_1 v_2 ... v_\ell$. Likewise, $\hat{\delta}(q, u)$ is the state the DFA will be in after reading $u_1, u_2, ..., u_k$ when starting from q. The DFA will be in state $\hat{\delta}(\hat{\delta}(q, u), v)$ after reading $v_1, v_2 ..., v_\ell$ additionally. Hence the claim follows.

   Or we can formally prove this claim by showing that
   $\hat{\delta}(q, uv) = \hat{\delta}(...(\hat{\delta}(\hat{\delta}(....(\hat{\delta}(q, u_1), ...), u_k), v_1), ...,)v_\ell) = \hat{\delta}(\hat{\delta}(q, u), v)$.

   See the lecture slides for the "most" formal proof.

2. Proving what strings a given DFA accepts.

   As an illustration, we consider the following very easy DFA $M = (Q = \{q_0, q_1\}, \{0, 1\}, \delta, q_0, \{q_0\})$ where $\delta(q_0, 0) = q_1$, $\delta(q_0, 1) = q_0$, $\delta(q_1, 0) = q_0$, and $\delta(q_1, 1) = q_1$. Give the state diagram of M. Do you see that M accepts all and only the strings with an even number of '0's. How can we prove it by induction? We give two proofs.

Proof:[The first proof] We would like to prove the following claim:

(a) $\hat{\delta}(q_0, w) = q_0$ iff w has an even number of '0's.

(b) $\hat{\delta}(q_0, w) = q_1$ iff w has an odd number of '0's.

We prove this claim by an induction on the length of w.

Base case $|w| = 0$, i.e. $w = \epsilon$.
In (a), both the left and right statements are true. In (b), both the left and right statements are false. So the claim is true when $|w| = 0$.

Induction step. Assuming that the claim is true when $|w| = k$, we would like to show the claim also holds when $|w| = k + 1$.

Towards this end, consider any string w with $|w| = k + 1$. Let $w = ua$ where $a \in \Sigma$ is a symbol. Note that $|u| = k$.

We consider two cases. The first case is when $a = 0$. Then, we have,

$$\begin{aligned} & \hat{\delta}(q_0, w = u0) = q_0 \\ \Leftrightarrow \quad & \hat{\delta}(q_0, u) = q_1 \qquad \text{[Since } \hat{\delta}(q_0, w = u0) = \delta(\hat{\delta}(q_0, u), 0), \text{ and } \delta(q, 0) = q_0 \text{ iff } q = q_1. \\ \Leftrightarrow \quad & u \text{ is a string with an odd number of '0's.} \qquad \text{[By induction hypothesis and knowing that } |u| = k.] \\ \Leftrightarrow \quad & w = u0 \text{ is a string with an even number of '0's.} \end{aligned}$$

Similarly, we have

$$\begin{aligned} & \hat{\delta}(q_0, w = u0) = q_1 \\ \Leftrightarrow \quad & \hat{\delta}(q_0, u) = q_0 \\ \Leftrightarrow \quad & u \text{ is a string with an even number of '0's.} \\ \Leftrightarrow \quad & w = u0 \text{ is a string with an odd number of '0's.} \end{aligned}$$

The proof for the case when $a = 1$ can be done similalrly, and is left as an exercise. That is, try to show

(a) $\hat{\delta}(q_0, w) = q_0$ iff w has an even number of '0's.

(b) $\hat{\delta}(q_0, w) = q_1$ iff w has an odd number of '0's.

when $a = 1$. □

Before moving to the second proof, let's take a close look at what is happening here.What the proof shows is that the above characterization stated in the claim remains true no matter how many more symbols the DFA reads. Suppose the following is true:

(a) $\hat{\delta}(q_0, w) = q_0$ iff w has an even number of '0's.

(b) $\hat{\delta}(q_0, w) = q_1$ iff w has an odd number of '0's.

In other words, this characterization shows the state the DFA will be in after reading string w depending on the number of '0's in w. So to put this simply, we have the following relation:
(q_0, the number of '0's we have read so far is even)
(q_1, the number of '0's we have read so far is odd)

If the DFA reads '0' now, what happens? The first relation becomes the second, and the second becomes the first. If the DFA reads '1' now, what happens? The first relation stays the same, likewise the second stays the same. So no new relations were created! It is straightforward to check the base case when $|w| = 0$.

Proof:[The second proof]

We would like to prove the following claim:

(a) $\hat{\delta}(q_0, w) \in F$ iff w has an even number of '0's.

(b) $\hat{\delta}(q_1, w) \in F$ iff w has an odd number of '0's.

We prove this claim by an induction on the length of w.

Base case $|w| = 0$, i.e. $w = \epsilon$.
In (a), both the left and right statements are true. In (b), both the left and right statements are false. So the claim is true when $|w| = 0$.

Induction step. Assuming that the claim is true when $|w| = k$, we would like to show the claim also holds when $|w| = k + 1$.

Towards this end, consider any string w with $|w| = k + 1$. Let $w = au$ where $a \in \Sigma$ is a symbol. Note that $|u| = k$.

We consider two cases. The first case is when $a = 0$. Then, we have,

$$\begin{array}{ll} & \hat{\delta}(q_0, 0u) \in F \\ \Leftrightarrow & \hat{\delta}(q_1, u) \in F \qquad \text{[Since } \hat{\delta}(q_0, 0u) = \hat{\delta}(q_1, u)\text{]} \\ \Leftrightarrow & u \text{ is a string with an odd number of '0's.} \qquad \text{[By induction hypothesis and knowing that } |u| = k.\text{]} \\ \Leftrightarrow & w = 0u \text{ is a string with an even number of '0's.} \end{array}$$

Similarly, we have,

$$\begin{array}{ll} & \hat{\delta}(q_1, 0u) \in F \\ \Leftrightarrow & \hat{\delta}(q_0, u) \in F \\ \Leftrightarrow & u \text{ is a string with an even number of '0's.} \\ \Leftrightarrow & w = 0u \text{ is a string with an odd number of '0's.} \end{array}$$

The proof of the other case when $a = 1$ is similar, and is left as an exercise.

□

Although the first proof seems more natural (at least to me), the second proof is slighlty easier to argue, so I recommend the second proof. The first proof shows how the DFA's current state evolves as it read the input strating from the initial state. The second proof shows how the DFA can transit from each state to one of the final states.