CSE 135: Introduction to Theory of Computation
Nondeterministic Finite Automata

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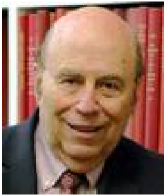
Nondeterminism

Michael Rabin and Dana Scott (1959)



Michael Rabin

Dana Scott

Nondeterminism

Given a current state of the machine and input symbol to be read, the next state is not uniquely determined.

Comparison to DFAs

Nondeterministic Finite Automata (NFA)

NFAs have 3 features when compared with DFAs.

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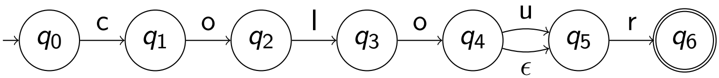
1. Ability to take a step without reading any input symbol
2. A state may have no transition on a particular symbol
3. Ability to transition to more than one state on a given symbol

ϵ-Transitions

Transitions without reading input symbols

Example

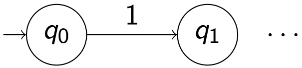
The British spelling of "color" is "colour". In a web search application, you may want to recognize both variants.



NFA with ϵ-transitions

No transitions

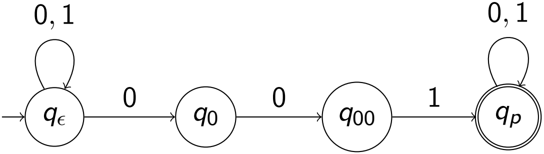
Example



No 0-transition out of initial state

In the above automaton, if the string starts with a 0 then the string has no computation (i.e., rejected).

Multiple Transitions



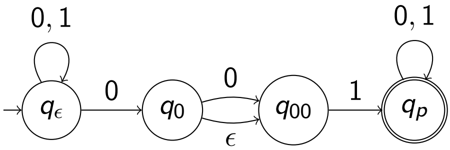
q_ϵ has two 0-transitions

Parallel Computation View

At each step, the machine "forks" a thread corresponding to one of the possible next states.

- If a state has an ϵ-transition, then you fork a new process for each of the possible ϵ-transitions, without reading any input symbol
- If the state has multiple transitions on the current input symbol read, then fork a process for each possibility
- If from current state of a thread, there is no transition on the current input symbol then the thread dies

Parallel Computation View: An Example



Example NFA

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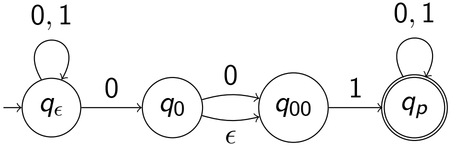
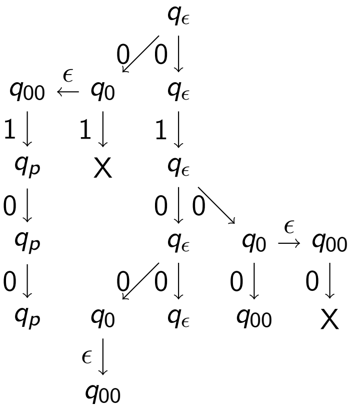


Example NFA



Computation on 0100

Nondeterministic Acceptance

Parallel Computation View

Input is accepted if after reading all the symbols, one of the live threads of the automaton is in a final/accepting state.

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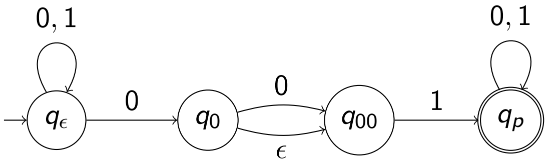

0100 is accepted because one thread of computation is
$q_\epsilon \xrightarrow{0} q_0 \xrightarrow{\epsilon} q_{00} \xrightarrow{1} q_p \xrightarrow{0} q_p \xrightarrow{0} q_p$

Computation: Guessing View

The machine magically guesses the choices that lead to acceptance

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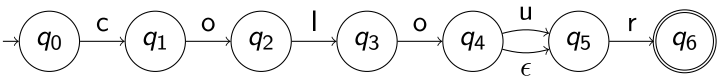
Example



NFA M_{color}

After seeing "colo" the automaton guesses if it will see the british or the american spelling. If it guesses american then it moves without reading the next input symbol.

Observations: Guessing View

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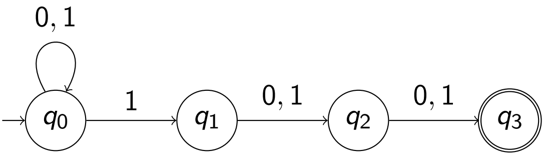
- On the other hand, if the input will not be accepted then no guess will lead the to automaton being in an accept state
  - On the input "colobr", whether automaton M_{color} guesses british or american, it will not proceed when it reads 'b'.

Example I



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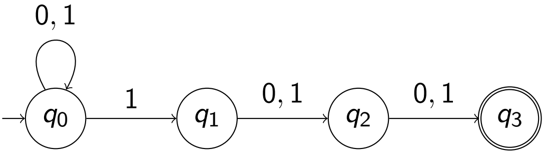
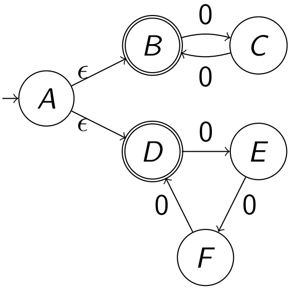


Automaton accepts strings having a 1 two positions from end of input

The automaton “guesses” at some point that the 1 it is seeing is 2 positions from end of input.

Example II



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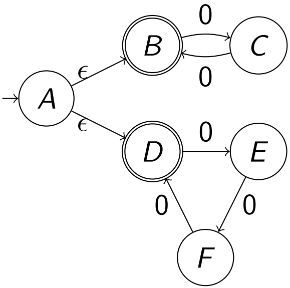
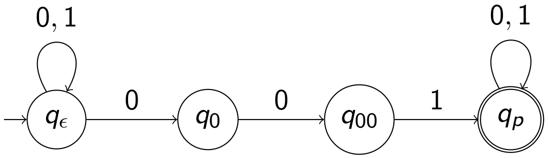


NFA accepting strings of 0s where the length is either a multiple 2 or 3

The NFA "guesses" at the begining whether it will see a multiple of 2 or 3, and then confirms that the guess was correct.

Example III



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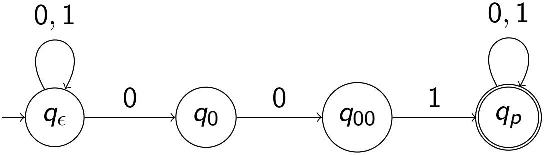

NFA accepting strings with 001 as substring

At some point the NFA "guesses" that the pattern 001 is starting and then checks to confirm the guess.

Nondeterministic Finite Automata (NFA)

Formal Definition

Definition

A nondeterministic finite automaton (NFA) is $M = (Q, \Sigma, \delta, q_0, F)$, where

- Q is the finite set of states
- Σ is the finite alphabet
- δ :
- $q_0 \in Q$ initial state
- $F \subseteq Q$ final/accepting states

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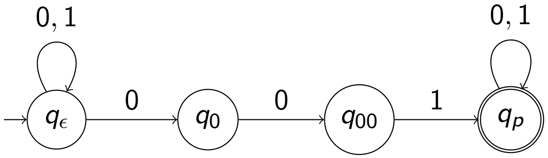
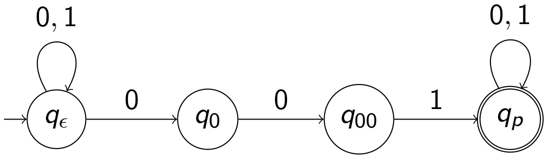
- Σ is the finite alphabet
- $\delta : Q \times (\Sigma \cup \{\epsilon\}) \to \mathcal{P}(Q)$, where $\mathcal{P}(Q)$ is the powerset of Q
- $q_0 \in Q$ initial state
- $F \subseteq Q$ final/accepting states

Example of NFA



Transition Diagram of NFA

Example of NFA



Transition Diagram of NFA

Formally, the NFA is $M_{001} = (\{q_\epsilon, q_0, q_{00}, q_p\}, \{0, 1\}, \delta, q_\epsilon, \{q_p\})$ where δ is given by

$$\begin{array}{lll} \delta(q_\epsilon, 0) = \{q_\epsilon, q_0\} & \delta(q_\epsilon, 1) = \{q_\epsilon\} & \delta(q_0, 0) = \{q_{00}\} \\ \delta(q_{00}, 1) = \{q_p\} & \delta(q_p, 0) = \{q_p\} & \delta(q_p, 1) = \{q_p\} \end{array}$$

δ is $\emptyset$ in all other cases.

Computation

Definition

For an NFA $M = (Q, \Sigma, \delta, q_0, F)$, string w, and states $q_1, q_2 \in Q$, we say $q_1 \xrightarrow{w}_M q_2$ if there is one thread of computation on input w from state q_1 that ends in q_2.

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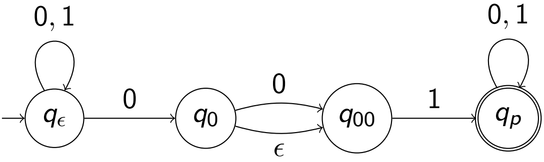
- $r_0 = q_1$,
- for each i, $r_{i+1} \in \delta(r_i, x_{i+1})$,
- $r_k = q_2$, and
- $w = x_1 x_2 x_3 \cdots x_k$

Example Computation



$q_\epsilon \xrightarrow{0100}_M q_p$ because taking $r_0 = q_\epsilon$, $r_1 = q_0$, $r_2 = q_{00}$, $r_3 = q_p$, $r_4 = q_p$, $r_5 = q_p$, and $x_1 = 0$, $x_2 = \epsilon$, $x_3 = 1$, $x_4 = 0$, $x_5 = 0$, we have

- $x_1 x_2 \cdots x_5 = 0\epsilon100 = 0100$
- $r_{i+1} \in \delta(r_i, x_{i+1})$

Defining $\hat{\Delta}$

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For an NFA $M = (Q, \Sigma, \delta, q_0, F)$, string w, and state $q_1 \in Q$, we say $\hat{\Delta}(q_1, w)$ to denote states of all the active threads of computation on input w from q_1.

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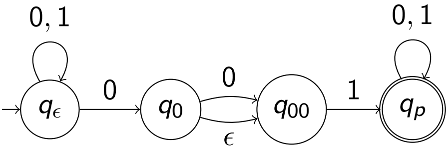
$$\hat{\Delta}(q_1, w) = \{q \in Q \mid q_1 \xrightarrow{w}_M q\}$$

Example



Example NFA

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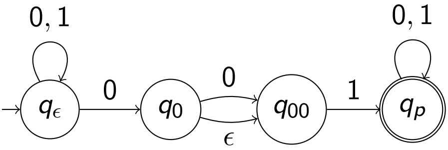

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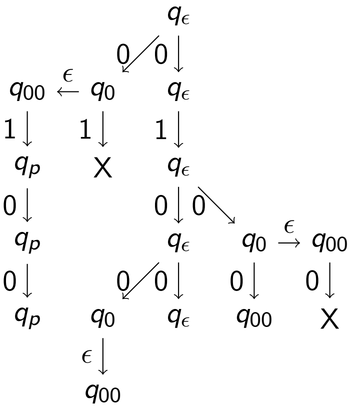

Computation on 0100

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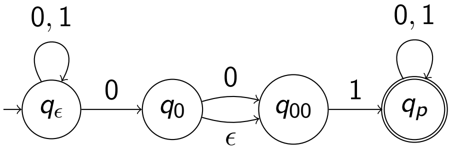
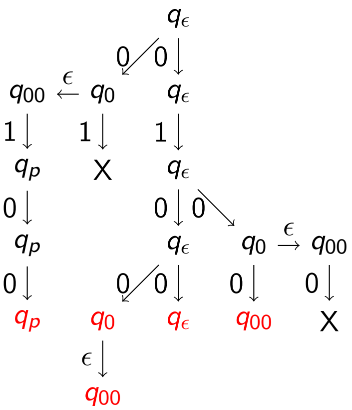


Example NFA

$$\hat{\Delta}(q_\epsilon, 0100) = \{q_0, q_p, q_{00}, q_\epsilon\}$$



Computation on 0100

Acceptance/Recognition

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For an NFA $M = (Q, \Sigma, \delta, q_0, F)$ and string $w \in \Sigma^*$, we say M accepts w iff $\hat{\Delta}(q_0, w) \cap F \neq \emptyset$

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The language accepted or recognized by NFA M over alphabet Σ is $L(M) = \{w \in \Sigma^* \mid M \text{ accepts } w\}$.

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However, the following proposition about DFAs continues to hold for NFAs

- For NFA M, strings u and v, and state q,

$$\hat{\Delta}(q, uv) = \bigcup_{q' \in \hat{\Delta}(q,u)} \hat{\Delta}(q', v)$$

Using Nondeterminism

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- But now, the machine can "guess" at certain steps

Back to the Future

Problem

For $\Sigma = \{0, 1, 2, \#\}$, let

$$L = \{w\#c \mid w \in \Sigma^*,\ c \in \Sigma,\ \text{and } c \text{ occurs in } w\}$$

So $1011\#0 \in L$ but $1011\#2 \notin L$. Design an NFA recognizing L.

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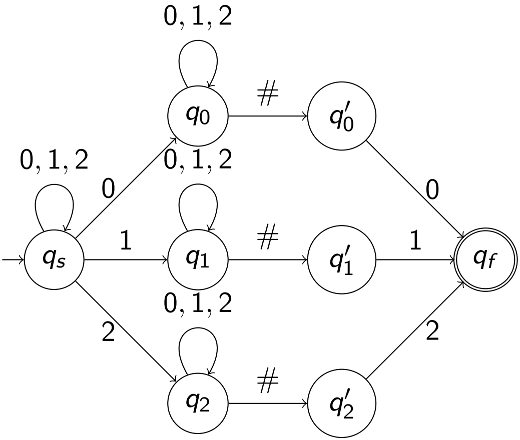
Solution

- Read symbols of w, i.e., portion of input before $\#$ is seen
- Guess at some point that current symbol in w is going to be the same as 'c'; store this symbol in the state
- Read the rest of w
- On reading $\#$, check that the symbol immediately after is the one stored, and that the input ends immediately after that.

Back to the Future

The Automaton



$$L(M) = \{w\#c \mid c \text{ occurs in } w\}$$

Halving a Language

Definition

For a language L, define $\frac{1}{2}L$ as follows.

$$\frac{1}{2}L = \{x \mid \exists y.\ |x| = |y| \text{ and } xy \in L\}$$

In other words, $\frac{1}{2}L$ consists of the first halves of strings in L

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If $L = \{001, 0000, 01, 110010\}$ then $\frac{1}{2}L = \qquad\qquad\qquad.$

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Proposition

If L is recognized by a DFA M then there is a NFA N such that $L(N) = \frac{1}{2}L$.

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- "Run" M on input x; let M be in state q_i after reading all of x
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How do you guess a string y of equal length to x using finite memory? Seems to require remembering the length of x!

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- How do you "run" M on y from q_i, if you cannot store all the symbols of y? Run M on y as you guess each symbol, without waiting to finish the execution on x!
- If we don't first execute M on x, how do we know the state q_i from which we have to execute y from? Guess it! And then check that running M on x does indeed end in q_i, your guessed state.

New Algorithm

On input x, NFA N

1. Guess state q_i and place "left finger" on (initial state of M) q_0 and "right finger" on q_i
2. As characters of x are read, N moves the left finger along transitions dictated by x and simultaneously moves the right finger along nondeterministically chosen transitions labelled by some symbol

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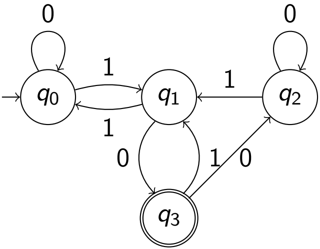
Things to remember: initial guess for right finger, and positions of left and right finger.

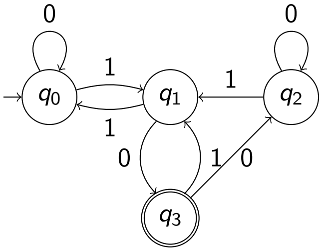
Algorithm on Example

$100010 \in L$ and so $x = 100 \in \frac{1}{2}L$



DFA M

Algorithm on Example



DFA M

$100010 \in L$ and so $x = 100 \in \frac{1}{2}L$
NFA N execution on $x = 100$ is

String Read	Left Finger	Right Finger
ϵ	q_0	q_2
1	q_1	q_2
10	q_3	q_1
100	q_2	q_3

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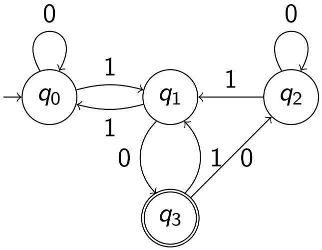

DFA M

String Read	Left Finger		Right Finger
ϵ	q_0		$\boxed{q_2}$
1	q_1		q_2
10	q_3	$=?$	q_1
100	$\boxed{q_2}$		$\boxed{q_3}$

$\uparrow$
accept?

Formal Construction of NFA N

States and Initial State

Given $M = (Q, \Sigma, \delta, q_0, F)$ recognizing L define $N = (Q', \Sigma, \delta', q_0', F')$ that recognizes $\frac{1}{2}L$

- $Q' = Q \times Q \times Q \cup \{s\}$, where $s \notin Q$
  - s is a new start state
  - Other states are of the form $\langle$left finger, initial guess, right finger$\rangle$; "initial guess" records the initial guess for the right finger
- $q_0' = s$

Formal Construction of NFA N

Transitions and Final States

- Transitions

  $$\delta'(s, \epsilon) = \{\langle q_0, q_i, q_i\rangle \mid q_i \in Q\}$$

  "Guess" the state q_i that the input will lead to

  $$\delta'(\langle q_i, q_j, q_k\rangle, a) = \{\langle q_l, q_j, q_m\rangle \mid \delta(q_i, a) = q_l, \\ \exists b \in \Sigma.\ \delta(q_k, b) = q_m\}$$

  b is the guess for the next symbol of y and initial guess does not change

- $F' = \{\langle q_i, q_i, q_j\rangle \mid q_i \in Q, q_j \in F\}$