CSE 135: Introduction to Theory of Computation
Rice's Theorem and Closure Properties

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Mapping Reductions

Definition

A function $f : \Sigma^* \to \Sigma^*$ is computable if there is some Turing Machine M that on every input w halts with $f(w)$ on the tape.

Definition

A reduction (a.k.a. mapping reduction/many-one reduction) from a language A to a language B is a computable function $f : \Sigma^* \to \Sigma^*$ such that

$$w \in A \text{ if and only if } f(w) \in B$$

In this case, we say A is reducible to B, and we denote it by $A \leq_m B$.

Reductions and Recursive Enumerability

Proposition

If $A \leq_m B$ and B is r.e., then A is r.e.

Proof.

Let f be a reduction from A to B and let M_B be a Turing Machine recognizing B. Then the Turing machine recognizing A is

```
On input w
    Compute f(w)
    Run M_B on f(w)
    Accept if M_B accepts, and reject if M_B rejects
```

$\square$

Corollary

If $A \leq_m B$ and A is not r.e., then B is not r.e.

Reductions and Decidability

Proposition

If $A \leq_m B$ and B is decidable, then A is decidable.

Proof.

Let f be a reduction from A to B and let M_B be a Turing Machine *deciding* B. Then a Turing machine that decides A is

On input w
    Compute $f(w)$
    Run M_B on $f(w)$
    Accept if M_B accepts, and reject if M_B rejects $\square$

Corollary

If $A \leq_m B$ and A is undecidable, then B is undecidable.

The Halting Problem

Proposition

The language $HALT = \{\langle M, w\rangle \mid M \text{ halts on input } w\}$ *is undecidable.*

Proof.

Recall $A_{\mathrm{TM}} = \{\langle M, w\rangle \mid w \in L(M)\}$ is undecidable. Will give reduction f to show $A_{\mathrm{TM}} \leq_m \text{HALT} \implies \text{HALT}$ undecidable.
Let $f(\langle M, w\rangle) = \langle N, w\rangle$ where N is a TM that behaves as follows:

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On input x
    Run M on x
    If M accepts then halt and accept
    If M rejects then go into an infinite loop
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N halts on input w if and only if M accepts w.

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On input x
    Run M on x
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N halts on input w if and only if M accepts w. i.e., $\langle M, w\rangle \in A_{\mathrm{TM}}$ iff $f(\langle M, w\rangle) \in \text{HALT}$ □

Emptiness of Turing Machines

Proposition

The language $E_{\mathrm{TM}} = \{M \mid L(M) = \emptyset\}$ *is not decidable.*

Note: in fact, E_{TM} is not recognizable.

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On input x

  If $x \neq w$, reject

  else run M on w , and accept if M accepts w

, and accept if B rejects $\langle M_1\rangle$, and rejects if B accepts $\langle M_1\rangle$.

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Then we show that (1) if $\langle M, w\rangle \in A_{\mathrm{TM}}$, then accept, and (2) $\langle M, w\rangle \in A_{\mathrm{TM}}$, then reject. (how?)

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Then we show that (1) if $\langle M, w\rangle \in A_{\mathrm{TM}}$, then accept, and (2) $\langle M, w\rangle \in A_{\mathrm{TM}}$, then reject. (how?) This implies A_{TM} is decidable, which is a contradiction. $\square$

Checking Regularity

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We give a reduction f from A_{TM} to REGULAR. Let $f(\langle M, w \rangle) = N$, where N is a TM that works as follows:

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On input x
    If x is of the form 0^n1^n then accept x
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If $w \in L(M)$ then $L(N) = \Sigma^*$. If $w \notin L(M)$ then $L(N) = \{0^n1^n \mid n \geq 0\}$. Thus, $\langle N\rangle \in$ REGULAR if and only if $\langle M, w\rangle \in A_{\mathrm{TM}}$ $\square$

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Checking Properties

Given M

$$\left.\begin{array}{l}\text{Does } L(M) \text{ contain } M?\\ \text{Is } L(M) \text{ non-empty?}\\ \text{Is } L(M) \text{ empty?}\end{array}\right\}\ \text{Undecidable}$$

Is $L(M)$ infinite?
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Is $L(M)$ co-finite (i.e., is $\overline{L(M)}$ finite)?
Is $L(M) = \Sigma^*$?

Which of these properties can be decided?

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Which of these properties can be decided? None!

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Which of these properties can be decided? None! By Rice's Theorem

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A *property of languages* is simply a set of languages.

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For any property $\mathbb{P}$, define language $L_{\mathbb{P}}$ to consist of Turing Machines which accept a language in $\mathbb{P}$:

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- Example: $\{M \mid L(M) \text{ is infinite}\}$; $E_{\mathrm{TM}} = \{M \mid L(M) = \emptyset\}$
- Non-example: $\{M \mid M \text{ has 15 states}\}$ $\longleftarrow$ This is a property of TMs, and not languages!

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Observation. For any trivial property $\mathbb{P}$, $L_{\mathbb{P}}$ is decidable. (Why?) Then $L_{\mathbb{P}} = \Sigma^*$ or $L_{\mathbb{P}} = \emptyset$.

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We cannot algorithmically determine any interesting property of languages represented as Turing Machines!

Properties of TMs

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Example

$$\left.\begin{array}{l}\{\langle M\rangle \mid M \text{ has 193 states}\}\\ \{\langle M\rangle \mid M \text{ uses at most 32 tape cells on blank input}\}\end{array}\right\}\ \text{Decidable}$$

$$\left.\begin{array}{l}\{\langle M\rangle \mid M \text{ halts on blank input}\}\\ \{\langle M\rangle \mid \text{ on input 0011 } M \text{ at some point writes the symbol \$ on its tape}\}\end{array}\right\}\ \text{Undecidable}$$

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Will show a reduction f that maps an instance $\langle M, w \rangle$ for A_{TM}, to N such that

- If M accepts w then N accepts the same language as M_0.
  - Then $L(N) = L(M_0) \in \mathbb{P}$
- If M does not accept w then N accepts $\emptyset$.
  - Then $L(N) = \emptyset \notin \mathbb{P}$

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Thus, $\langle M, w \rangle \in A_{\mathrm{TM}}$ iff $N \in L_{\mathbb{P}}$.

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Proof (contd).

The reduction f maps $\langle M, w\rangle$ to N, where N is a TM that behaves as follows:

```
On input x
    Ignore the input and run M on w
    If M does not accept (or doesn't halt)
        then do not accept x (or do not halt)
    If M does accept w
        then run M_0 on x and accept x iff M_0 does.
```

Notice that indeed if M accepts w then $L(N) = L(M_0)$. Otherwise $L(N) = \emptyset$. $\square$

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- Rice's theorem says nothing about properties of Turing machines

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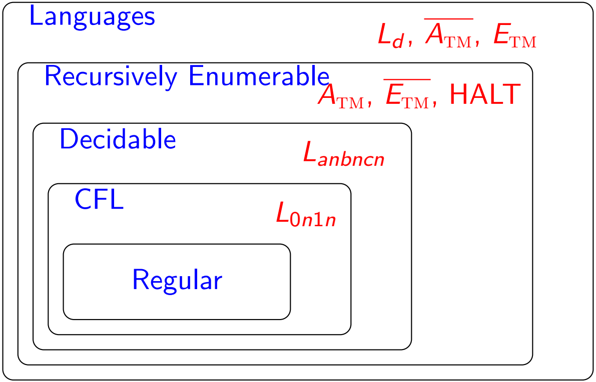
Recap

Every non-trivial property of r.e. languages is undecidable

- Rice's theorem says nothing about properties of Turing machines
- Rice's theorem says nothing about whether a property of languages is recurisvely enumerable or not.

Big Picture . . . again



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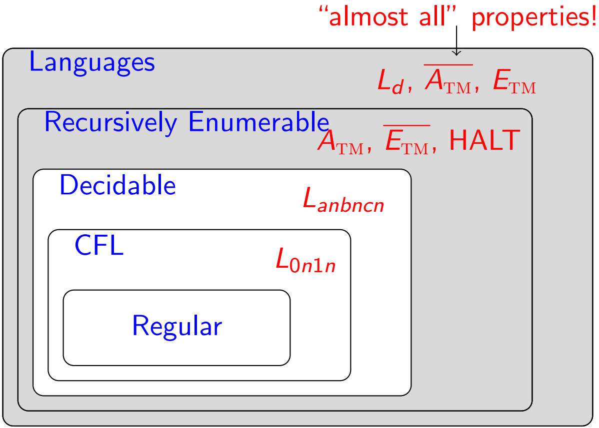

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Decidable languages are closed under union, intersection, and complementation.

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Proof.

Given TMs M_1, M_2 that decide languages L_1, and L_2

- A TM that decides $L_1 \cup L_2$: on input x, run M_1 and M_2 on x, and accept iff either accepts.

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- A TM that decides $\overline{L_1}$: On input x, run M_1 on x, and accept if M_1 rejects, and reject if M_1 accepts. $\square$

Regular Operators

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Decidable languages are closed under concatenation and Kleene Closure.

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- A TM to decide L_1^*: On input x, if $x = \epsilon$ accept. Else, for each of the $2^{|x|-1}$ ways to divide x as $w_1 \dots w_k$ $(w_i \neq \epsilon)$: run M_1 on each w_i and accept if M_1 accepts all. Else reject. $\square$

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Complementation

Proposition

R.E. languages are not closed under complementation.

Proof.

A_{TM} is r.e. but $\overline{A_{\mathrm{TM}}}$ is not. $\square$

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