CSE 135: Introduction to Theory of Computation
Decidability and Recognizability

Sungjin Im

University of California, Merced

04-14 and 4-16-2015

High-Level Descriptions of Computation

- Instead of giving a Turing Machine, we shall often describe a program as code in some programming language (or often "pseudo-code")
  - Possibly using high level data structures and subroutines (Recall that TM and RAM are equivalent (even polynomially))
- Inputs and outputs are complex objects, encoded as strings
- Examples of objects:
  - Matrices, graphs, geometric shapes, images, videos, ...
  - DFAs, NFAs, Turing Machines, Algorithms, other machines ...

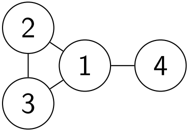
High-Level Descriptions of Computation

Encoding Complex Objects

- "Everything" finite can be encoded as a (finite) string of symbols from a finite alphabet (e.g. ASCII)
  - Can in turn be encoded in binary (as modern day computers do). No special $\sqcup$ symbol: use self-terminating representations
- Example: encoding a "graph."

  `(1,2,3,4)((1,2)(2,3)(3,1)(1,4))`

  encodes the graph



High-Level Descriptions of Computation

- We have already seen several algorithms, for problems involving complex objects like DFAs, NFAs, regular expressions, and Turing Machines
  - For example, convert a NFA to DFA; Given a NFA N and a word w, decide if $w \in L(N)$; ...
- All these inputs can be encoded as strings and all these algorithms can be implemented as Turing Machines
- Some of these algorithms are for decision problems, while others are for computing more general functions
- All these algorithms terminate on all inputs

High-Level Descriptions of Computation

Examples: Problems regarding Computation

Some more decision problems that have algorithms that always halt (sketched in the textbook)

- On input $\langle B, w \rangle$ where B is a DFA and w is a string, decide if B accepts w.
  Algorithm: simulate B on w and accept iff simulated B accepts
- On input $\langle B \rangle$ where B is a DFA, decide if $L(B) = \emptyset$.
  Algorithm: Use a fixed point algorithm to find all reachable states. See if any final state is reachable.

Code is just data: A TM can take "the code of a program" (DFA, NFA or TM) as part of its input and analyze or even execute this code

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Universal Turing Machine (a simple "Operating System"): Takes a TM M and a string w and simulates the execution of M on w

Decidable and Recognizable Languages

Recall: Definition

A Turing machine M is said to recognize a language L if $L = L(M)$.
A Turing machine M is said to decide a language L if $L = L(M)$ and M halts on every input.

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L is said to be Turing-recognizable (Recursively Enumerable (R.E.) or simply recognizable) if there exists a TM M which recognizes L.
L is said to be Turing-decidable (Recursive or simply decidable) if there exists a TM M which decides L.

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- Every finite language is decidable: For example, by a TM that has all the strings in the language "hard-coded" into it
- We just saw some example algorithms all of which terminate in a finite number of steps, and output yes or no (accept or reject). i.e., They decide the corresponding languages.

Decidable and Recognizable Languages

- But not all languages are decidable! We will show:
  - $A_{\text{TM}} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$ is undecidable

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- However A_{TM} is Turing-recognizable!

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- However A_{TM} is Turing-recognizable!

Proposition

There are languages which are recognizable, but not decidable

Recognizing A_{TM}

Program U for recognizing A_{TM}:

On input $\langle M, w \rangle$
    simulate M on w
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But U does not decide A_{TM}: If M rejects w by not halting (does not halt on w), U rejects $\langle M, w \rangle$ by not halting (does not halt on $\langle M, w \rangle$).

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But U does not decide A_{TM}: If M rejects w by not halting (does not halt on w), U rejects $\langle M, w\rangle$ by not halting (does not halt on $\langle M, w\rangle$). Indeed (as we shall see) no TM decides A_{TM}.

Deciding vs. Recognizing

Proposition

If L and $\overline{L}$ are recognizable, then L is decidable

Proof.

Program P for deciding L, given programs P_L and $P_{\overline{L}}$ for recognizing L and $\overline{L}$:

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- On input x, simulate P_L and $P_{\overline{L}}$ on input x. Whether $x \in L$ or $x \notin L$, one of P_L and $P_{\overline{L}}$ will halt in finite number of steps.
- Which one to simulate first?

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- Which one to simulate first? Either could go on forever.
- On input x, simulate in parallel P_L and $P_{\overline{L}}$ on input x until either P_L or $P_{\overline{L}}$ accepts

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- Which one to simulate first? Either could go on forever.
- On input x, simulate in parallel P_L and $P_{\overline{L}}$ on input x until either P_L or $P_{\overline{L}}$ accepts
- If P_L accepts, accept x and halt. If $P_{\overline{L}}$ accepts, reject x and halt. $\cdots\longrightarrow$

Deciding vs. Recognizing

Proof (contd).

In more detail, P works as follows:

```
On input x
for i = 1, 2, 3, ...
    simulate P_L on input x for i steps
    simulate P_L̄ on input x for i steps
    if either simulation accepts, break
if P_L accepted, accept x (and halt)
if P_L̄ accepted, reject x (and halt)
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Proof (contd).

In more detail, P works as follows:

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for i = 1, 2, 3, ...
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if P_L̄ accepted, reject x (and halt)
```

(Alternately, maintain configurations of P_L and $P_{\overline{L}}$, and in each iteration of the loop advance both their simulations by one step.) $\square$

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- But it is recognizable

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Note: Decidable languages are closed under complementation, but recognizable languages are not.

Decision Problems and Languages

- A decision problem requires checking if an input (string) has some property. Thus, a decision problem is a function from `strings` to `boolean`.
- A decision problem is represented as a formal language consisting of those strings (inputs) on which the answer is "yes".

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- A Turing Machine on an input w either (halts and) accepts, or (halts and) rejects, or never halts.
- The language of a Turing Machine M, denoted as $L(M)$, is the set of all strings w on which M accepts.
- A language L is recursively enumerable/Turing recognizable if there is a Turing Machine M such that $L(M) = L$.

Decidability

- A language L is decidable if there is a Turing machine M such that $L(M) = L$ and M halts on every input.

Decidability

- A language L is decidable if there is a Turing machine M such that $L(M) = L$ and M halts on every input.
- Thus, if L is decidable then L is recursively enumerable.

Undecidability

Definition
A language L is undecidable if L is not decidable.

Undecidability

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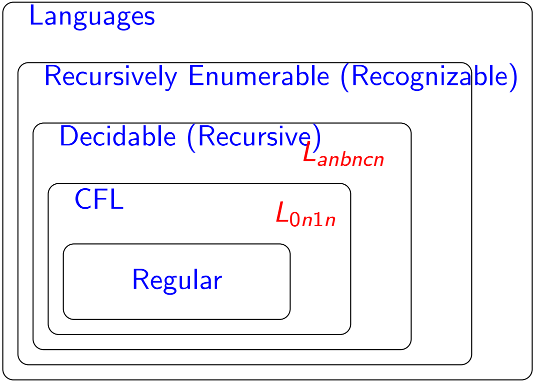
A language L is undecidable if L is not decidable. Thus, there is no Turing machine M that halts on every input and $L(M) = L$.

- This means that either L is not recursively enumerable. That is there is no turing machine M such that $L(M) = L$, or
- L is recursively enumerable but not decidable. That is, any Turing machine M such that $L(M) = L$, M does not halt on some inputs.

Big Picture



Relationship between classes of Languages

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- Any Turing Machine/program M can itself be encoded as a binary string. Moreover every binary string can be thought of as encoding a TM/program. (If not the correct format, considered to be the encoding of a default TM.)
- We will consider decision problems (language) whose inputs are Turing Machine (encoded as a binary string)

The Diagonal Language

Definition

Define $L_d = \{M \mid M \notin L(M)\}$.

The Diagonal Language

Definition

Define $L_d = \{M \mid M \notin L(M)\}$. Thus, L_d is the collection of Turing machines (programs) M such that M does not halt and accept (i.e. either reject or never ends) when given itself as input.

A non-Recursively Enumerable Language

Proposition

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Proof.

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- In what follows, we will denote the ith binary string (in lexicographic order) as the number i. Thus, we can say $j \in L(i)$, which means that the Turing machine corresponding to ith binary string accepts the jth binary string. $\cdots\longrightarrow$

Proof (contd).

We can organize all programs and inputs as a (infinite) matrix, where the (i, j)th entry is Y if and only if $j \in L(i)$.

Inputs $\longrightarrow$

TMs $\downarrow$		1	2	3	4	5	6	7	$\cdots$
	1	N	N	N	N	N	N	N	
	2	N	N	N	N	N	N	N	
	3	Y	N	Y	N	Y	Y	Y	
	4	N	Y	N	Y	Y	N	N	
	5	N	Y	N	Y	Y	N	N	
	6	N	N	Y	N	Y	N	Y	

Completing the proof

Diagonalization: Cantor

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For the sake of contradiction, suppose L_d is recognized by a Turing machine. Say by the jth binary string. i.e., $L_d = L(j)$.

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	4	N	Y	N	$\boxed{\text{Y}}$	Y	N	N	
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For the sake of contradiction, suppose L_d is recognized by a Turing machine. Say by the jth binary string. i.e., $L_d = L(j)$. But $j \in L_d$ iff $j \notin L(j)$! More concretly, suppose $j \notin L(j)$ – note that j can be a string or a TM. Then, by definition, $j \in L_d = L(j)$. The other case $j \in L(j)$ can be handled similarly. $\square$

Acceptor for L_d?

Consider the following program

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On input i
    Run program i on i
    Output ``yes'' if i does not accept i
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Does the above program recognize L_d? No, because it may never output "yes" if i does not halt on i.

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Recursively Enumerable but not Decidable

- L_d not recursively enumerable, and therefore not decidable. Are there languages that are recursively enumerable but not decidable?
- Yes, $A_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$

The Universal Language

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Proof.

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On input i
    Run M on input $\langle i, i \rangle$
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Observe that $L(D) = L_d$!

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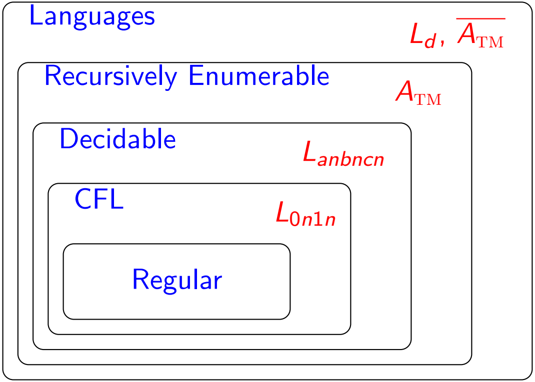
Observe that $L(D) = L_d$! But, L_d is not r.e. which gives us the contradiction. $\square$

A more complete Big Picture



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- Informal Examples: Measuring the area of rectangle reduces to measuring the length of the sides; Solving a system of linear equations reduces to inverting a matrix
- The problem L_d reduces to the problem A_{TM} as follows: "To see if $w \in L_d$ check if $\langle w, w \rangle \in A_{\mathrm{TM}}$."

Undecidability using Reductions

Proposition

Suppose L_1 *reduces to* L_2 *and* L_1 *is undecidable. Then* L_2 *is undecidable.*

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Suppose L_1 reduces to L_2 and L_1 is undecidable. Then L_2 is undecidable.

Proof Sketch.

Suppose for contradiction L_2 is decidable. Then there is a M that always halts and decides L_2. Then the following algorithm decides L_1

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Proof Sketch.

Suppose for contradiction L_2 is decidable. Then there is a M that always halts and decides L_2. Then the following algorithm decides L_1

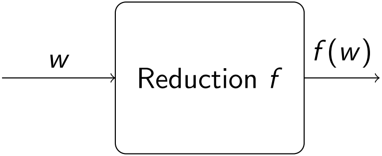
- On input w, apply reduction to transform w into an input w' for problem 2
- Run M on w', and use its answer.

Schematic View

$$\xrightarrow{w}$$

Reductions schematically

Schematic View



Reductions schematically

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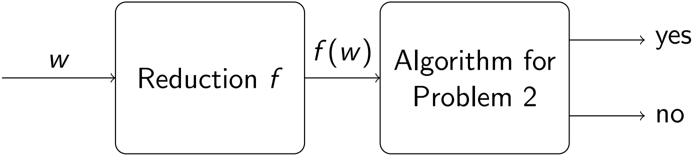
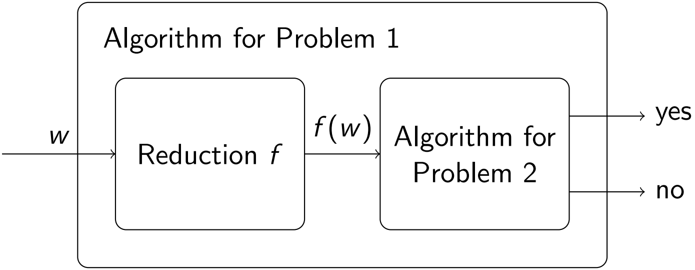


Reductions schematically

Schematic View



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The Halting Problem

Proposition

The language $HALT = \{\langle M, w\rangle \mid M \text{ halts on input } w\}$ *is undecidable.*

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On input x
    Run M on x
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Observe that $f(M)$ halts on input w if and only if M accepts w $\cdots\longrightarrow$

The Halting Problem

Completing the proof

Proof (contd).

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T decides A_{TM}.

Proof (contd).

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T decides A_{TM}. But, A_{TM} is undecidable, which gives us the contradiction. □

Mapping Reductions

Definition

A function $f : \Sigma^* \to \Sigma^*$ is computable if there is some Turing Machine M that on every input w halts with $f(w)$ on the tape.

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A mapping/many-one reduction from A to B is a computable function $f : \Sigma^* \to \Sigma^*$ such that

$$w \in A \text{ if and only if } f(w) \in B$$

In this case, we say A is mapping/many-one reducible to B, and we denote it by $A \leq_m B$.

Convention

In this course, we will drop the adjective "mapping" or "many-one", and simply talk about reductions and reducibility.

Reductions and Recursive Enumerability

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If $A \leq_m B$ *and* B *is recursively enumerable then* A *is recursively enumerable.*

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Let f be the reduction from A to B and let M_B be the Turing Machine recognizing B.

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Let f be the reduction from A to B and let M_B be the Turing Machine recognizing B. Then the Turing machine recognizing A is

```
On input w
    Compute f(w)
    Run M_B on f(w)
    Accept if M_B does and reject if M_B rejects
```

$\square$

Reductions and non-r.e.

Corollary

If $A \leq_m B$ and A is not recursively enumerable then B is not recursively enumerable.

Reductions and Decidability

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Let M_B be the Turing machine deciding B and let f be the reduction. Then the algorithm deciding A, on input w, computes $f(w)$ and runs M_B on $f(w)$. $\square$

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If $A \leq_m B$ and B is decidable then A is decidable.

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Let M_B be the Turing machine deciding B and let f be the reduction. Then the algorithm deciding A, on input w, computes $f(w)$ and runs M_B on $f(w)$. $\square$

Corollary

If $A \leq_m B$ and A is undecidable then B is undecidable.