CSE 135: Introduction to Theory of Computation
Variants of Turing Machines and Church-Turing Thesis

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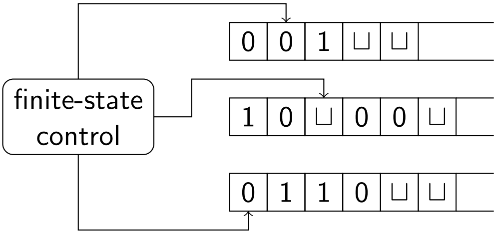
04-09-2015

Multi-Tape Turing Machine



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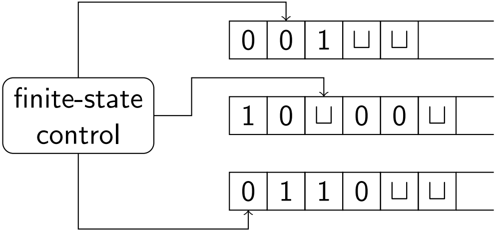

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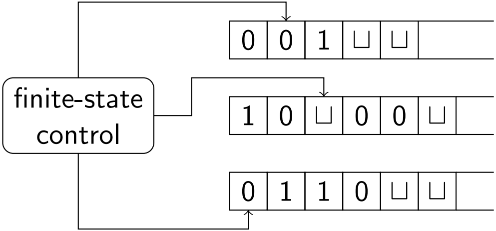

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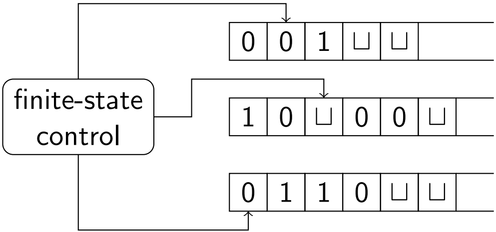

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- Initially all heads scanning cell 1, and tapes 2 to k blank
- In one step: Read symbols under each of the k-heads, and depending on the current control state, write new symbols on the tapes, move the each tape head (possibly in different directions), and change state.

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- $\delta : (Q \setminus \{q_{\mathsf{acc}}, q_{\mathsf{rej}}\}) \times \Gamma^k \to Q \times (\Gamma \times \{\mathsf{L}, \mathsf{R}\})^k$ is the transition function.

Computation, Acceptance and Language

- A configuration of a multi-tape TM must describe the state, contents of all k-tapes, and positions of all k-heads. Thus, $\mathrm{C} \in Q \times (\Gamma^*\{*\}\Gamma\Gamma^*)^k$, where $*$ denotes the head position.

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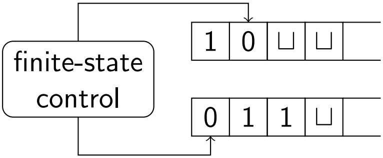
- How do we simulate the movement of k independent heads?

Storing Multiple Tapes



Multi-tape TM M

Store in cell i contents of cell i of all tapes.

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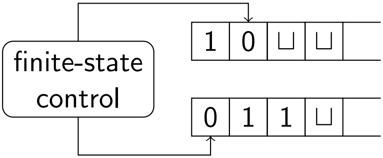

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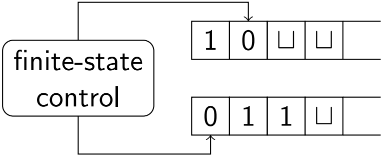

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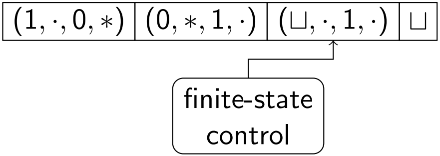

1-tape equivalent single(M)

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- Once again, scan the tape, change all relevant contents, "move" heads (i.e., move $*$s), and change state.

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    - Scan back from right-to-left moving the heads that need to be moved left.

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- Idea 2: det(M) will simulate M on each possible sequence of computation steps that M may try in each step.

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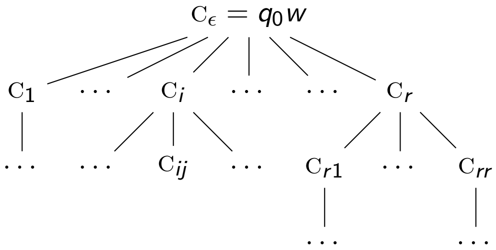

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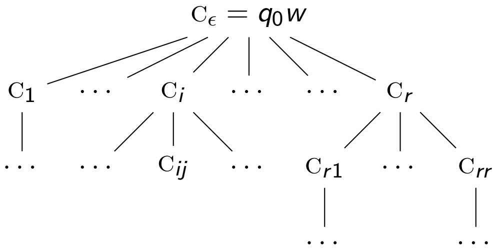

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- Input w is accepted iff $\exists$ accepting configuration in tree.

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Observe that det(M) may not terminate if w is not accepted.

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- Tape 3, called choice tape, will store the current sequence of nondeterministic choices

Execution of det(M)

1. Initially: Input tape contains w, simulation tape and choice tape are blank
2. Copy contents of input tape onto simulation tape
3. Simulate M using simulation tape as its (only) tape
    3.1 At the next step of M, if state is q, simulation tape head reads X, and choice tape head reads i, then simulate the ith possibility in $\delta(q, X)$; if i is not a valid choice, then goto step 4
    3.2 After changing state, simulation tape contents, and head position on simulation tape, move choice tape's head to the right. If Tape 3 is now scanning $\sqcup$, then goto step 4
    3.3 If M accepts then accept and halt, else goto step 3(1) to simulate the next step of M.
4. Write the lexicographically next choice sequence on choice tape, erase everything on simulation tape and goto step 2.

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- If M accepts w then there is a sequence of choices that will lead to acceptance. det(M) will eventually have this sequence on choice tape, and then its simulation M will accept.
- If M does not accept w then no sequence of choices leads to acceptance. det(M) will therefore never halt!

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- Initially, the program instructions are stored in a contiguous block of memory locations starting at location 1. All registers and memory locations, other than those storing the program, are set to 0.

Instruction Set

- `add X, Y`: Add the contents of registers X and Y and store the result in X.
- `loadc X, I`: Place the constant I in register X.
- `load X, M`: Load the contents of memory location M into register X.
- `loadI X, M`: Load the contents of the location "pointed to" by the contents of M into register X.
- `store X, M`: store the contents of register X in memory location M.
- `jmp M`: The next instruction to be executed is in location M.
- `jmz X, M`: If register X is 0, then jump to instruction M.
- `halt`: Halt execution.

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- Enhanced Turing Machine models: TM with 2-way infinite tape, multi-tape TM, nondeterministic TM, probabilistic Turing Machines, quantum Turing Machines ...
- Restricted Turing Machine models: queue machines, 2-stack machines, 2-counter machines, ...

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- Strong evidence based on the fact that many attempts to define computation yield the same expressive power

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- To show that something can be solved on Turing machines, you can use any programming language of choice, *unless the problem specifically asks you to design a Turing Machine*

Terminology for Describing Turing Machines

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2. Implmentation description. Use English prose to describe the way that the TM moves its head and the way that it stores data on its tape. Intermediate level of description. Note that you can only use "local" knowledge.

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3. High-level description. Describe an algorithm, ignoring implementation. Highest level of description. Justified by Church-Turing thesis.

Revisiting Type 0 grammar

Grammars

Definition

A grammar is $G = (V, \Sigma, R, S)$, where

- V is a finite set of variables/non-terminals
- Σ is a finite set of terminals
- $S \in V$ is the start symbol
- $R \subseteq (\Sigma \cup V)^* \times (\Sigma \cup V)^*$ is a finite set of rules/productions

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$L(G) = \{w \in \Sigma^* \mid S \overset{*}{\Rightarrow}_G w\}$

Example

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Consider the grammar G with $\Sigma = \{a\}$ with

$$\begin{array}{lll} S \rightarrow \$Ca\# \mid a \mid \epsilon & Ca \rightarrow aaC & \$D \rightarrow \$C \\ C\# \rightarrow D\# \mid E & aD \rightarrow Da & aE \rightarrow Ea \\ \$E \rightarrow \epsilon & & \end{array}$$

The following are derivations in this grammar

$$S \Rightarrow \$Ca\# \Rightarrow \$aaC\# \Rightarrow \$aaE \Rightarrow \$aEa \Rightarrow \$Eaa \Rightarrow aa$$

$$\begin{aligned} S & \Rightarrow \$Ca\# \Rightarrow \$aaC\# \Rightarrow \$aaD\# \Rightarrow \$aDa\# \Rightarrow \$Daa\# \Rightarrow \$Caa\# \\ & \Rightarrow \$aaCa\# \Rightarrow \$aaaaC\# \Rightarrow \$aaaaE \Rightarrow \$aaaEa \Rightarrow \$aaEaa \\ & \Rightarrow \$aEaaa \Rightarrow \$Eaaaa \Rightarrow aaaa \end{aligned}$$

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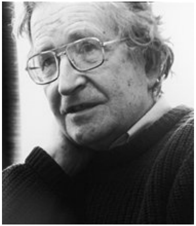
$L(G) = \{a^i \mid i \text{ is a power of } 2\}$

Grammars for each task

- What is the expressive power of these grammars?

Grammars for each task



Noam Chomsky

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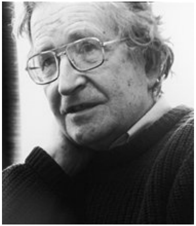

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- What is the expressive power of these grammars?
- Restricting the types of rules, allows one to describe different aspects of natural languages
- These grammars form a hierarchy

Type 0 Grammars

Definition

Type 0 grammars are those where the rules are of the form

$$\alpha \to \beta$$

where $\alpha, \beta \in (\Sigma \cup V)^*$

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Expressive Power of Type 0 Grammars

Theorem

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Thus, type 0 grammars are as powerful as Turing machines.

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Proposition

If $G = (V, \Sigma, R, S)$ is a type 0 grammar then $L(G)$ is recursively enumerable.

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Proof.

We will show that $L(G)$ is recognized by a 2-tape non-deterministic Turing machine M, with tape 1 storing the input w, and tape 2 used to construct a derivation of w from S. $\cdot\cdot\longrightarrow$

Recognizing Type 0 Grammars

Proof (contd).

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- To simulate the next derivation step, M will (nondeterministically) choose a rule to apply, scan from left to right and choose (nondeterministically) a position to apply the rule, replace the substring matching the LHS of the rule with the RHS to get the string at the next step of derivation.

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- If tape 2 contains only terminal symbols, then M will check to see if it matches tape 1. If so, the input is accepted, else it is rejected. □

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Let M be a Turing machine recognizing L. The grammar G will simulate M "backwards" starting from an accepting configuration.

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- The rules of S will generate an accepting configuration of M
- Once (some) initial configuration $q_0 w$ is generated, rules in G will erase symbols to produce the terminal w.

□