CSE 135: Introduction to Theory of Computation
Turing Machines

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  - Limitations on how much memory they can use: fixed amount of memory
  - Limitations on what they can compute/decide: only regular languages
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- The complete machine?
  - No limitations on memory usage? And maybe other ways to use computational resources that we haven’t thought of...
    - Come up with a model that describes all “conceivable” computation
  - No limitation on what they can compute?
    - No! There are far too many languages over \( \{0, 1\} \) than there are “machines” or programs (as long as machines can be represented digitally)
General Computing Machines
Alonzo Church, Emil Post, and Alan Turing (1936)

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- In this course: Turing Machines
The ‘aha’ moment
The ‘aha’ moment

GREAT. A WAREHOUSE FILLED WITH MILES AND MILES OF REWRITABLE TAPE! WHAT ARE WE EVER GOING TO DO WITH THIS, ALAN?

...ALAN?

And thus the Turing Machine was born.
Turing Machines

- Unrestricted memory: an infinite tape
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- Unrestricted memory: an infinite tape
  - A finite state machine that reads/writes symbols on the tape
  - Can read/write anywhere on the tape
  - Tape is infinite in one direction only (other variants possible)
- Initially, tape has input and the machine is reading (i.e., tape head is on) the leftmost input symbol.
- Transition (based on current state and symbol under head):
  - Change control state
  - Overwrite a new symbol on the tape cell under the head
  - Move the head left, or right.
A Turing machine is $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$ where

- $Q$ is a finite set of control states
- $\Sigma$ is a finite set of input symbols
- $\Gamma \supseteq \Sigma$ is a finite set of tape symbols, also a blank symbol $\_ \in \Gamma \setminus \Sigma$
- $q_0 \in Q$ is the initial state
- $q_{\text{acc}} \in Q$ is the accept state
- $q_{\text{rej}} \in Q$ is the reject state, where $q_{\text{rej}} \neq q_{\text{acc}}$
- $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$ is the transition function.

Given the current state and symbol being read, the transition function describes the next state, symbol to be written and direction (left or right) in which to move the tape head.
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Formal Definition

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Transition Function

\[ \delta(q_1, X) = (q_2, Y, L) \]: Read transition as “the machine when in state \( q_1 \), and reading symbol \( X \) under the tape head, will move to state \( q_2 \), overwrite \( X \) with \( Y \), and move its tape head to the left”
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- In fact, \( \delta : (Q \setminus \{q_{\text{acc}}, q_{\text{rej}}\}) \times \Gamma \rightarrow Q \times \Gamma \times \{\text{L, R}\} \).
Transition Function

\[ q_1 \xrightarrow{X \rightarrow Y, L} q_2 \]

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- Transitions are deterministic
- Convention: if \( \delta(q, X) \) is not explicitly specified, it is taken as leading to \( q_{\text{rej}} \), i.e., say \( \delta(q, X) = (q_{\text{rej}}, \Box, R) \)
Configurations

The configuration (or “instantaneous description”) contains all the information to exactly capture the “current state of the computation”
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Special Configurations

- **Start configuration**: $q_0X_1 \cdots X_n$, where the input is $X_1 \cdots X_n$
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- **Accept and reject configurations:** The state $q$ is $q_{\text{acc}}$ or $q_{\text{rej}}$, respectively. These configurations are halting configurations, because there are no transitions possible from them.
Single Step

Definition
We say one configuration \((c_1)\) yields another \((c_2)\), denoted as \(c_1 \vdash c_2\), if one of the following holds.

- If \(\delta(q, X_i) = (p, Y, L)\) then
  
  \[X_1X_2 \cdots X_{i-1}qX_iX_{i+1} \cdots X_n \vdash X_1X_2 \cdots X_{i-2}pX_{i-1}YX_{i+1} \cdots X_n\]
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Boundary Cases:

▶ If $i = 1$ then
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- If $\delta(q, X_i) = (p, Y, R)$ then

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Boundary Cases:
- If \(i = 1\) then \(qX_1 X_2 \cdots X_n \vdash pY X_2 \cdots X_n\)
- If \(i = n\) and \(Y = \square\) then \(X_1 \cdots X_{n-1} qX_n \vdash X_1 \cdots pX_{n-1}\)
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X_1 X_2 \cdots X_{i-1} qX_i X_{i+1} \cdots X_n \vdash X_1 X_2 \cdots X_{i-1} Y pX_{i+1} \cdots X_n
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  **Boundary Case:**
  - If \(i = n\) then \(X_1 \cdots X_{n-1}qX_n \vdash X_1 \cdots X_{n-1}Yp\square\)
Computations

Definition
We say $c_1 \vdash^* c_2$ if the machine can move from $c_1$ to $c_2$ in zero or more steps.
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Acceptance and Recognition

Definition
A Turing machine $M$ accepts $w$ iff $q_0 w \vdash *\alpha_1 q_{\text{acc}} \alpha_2$, where $\alpha_1, \alpha_2$ are some strings. In other words, the machine $M$ when started in its initial state and with $w$ as input, reaches the accept state.
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Note: The machine may not read all the symbols in $w$. It may pass back and forth over some symbols of $w$ several times. Finally, $w$ may have been completely overwritten.
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Definition
For a Turing machine $M$, define $L(M) = \{ w \mid M$ accepts $w \}$. $M$ is said to accept or recognize a language $L$ if $L = L(M)$. 
Example 1: TM for $\{0^n1^n \mid n > 0\}$

Design a TM to accept the language $L = \{0^n1^n \mid n > 0\}$
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High level description

On input string $w$

while there are unmarked 0s, do

Mark the left most 0
Scan right till the leftmost unmarked 1;
if there is no such 1 then crash
Mark the leftmost 1

done

Check to see that there are no unmarked 1s;
if there are then crash
accept
Example 1: TM for \( \{0^n1^n \mid n > 0\} \)
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- \( q_0 \) transitions:
  - \( 0 \to A, R \)
  - \( B \to B, R \)
- \( q_1 \) transitions:
  - \( 1 \to B, L \)
  - \( 0 \to 0, R \)
  - \( B \to B, R \)
- \( q_2 \) transitions:
  - \( B \to B, L \)
  - \( 0 \to 0, L \)
- \( q_3 \) transitions:
  - \( \square \to \square, R \)
- \( q_{\text{acc}} \) transitions:
  - \( B \to B, R \)

- Accepts input 0011: \( q_00011 \vdash \)
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- Accepts input 0011: \( q_00011 \vdash Aq_1011 \vdash \)

$$
\begin{align*}
q_0 & \quad 0 \rightarrow A, R \\
\rightarrow q_1 & \quad 1 \rightarrow B, L \\
\rightarrow q_2 & \quad B \rightarrow B, R \\
\rightarrow q_3 & \quad B \rightarrow B, R \\
\rightarrow q_{acc} & \quad B \rightarrow B, R \\
\end{align*}
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Example 1: TM for \( \{0^n1^n \mid n > 0\} \)

- \( q_0 \) to \( q_1 \): \( 0 \rightarrow A, R \)
- \( q_1 \) to \( q_2 \): \( 1 \rightarrow B, L \)
- \( q_2 \) to \( q_3 \): \( B \rightarrow B, L \)
- \( q_3 \) to \( q_{\text{acc}} \): \( \sqcup \rightarrow \sqcup, R \)

- Accepts input 0011: \( q_00011 \vdash Aq_1011 \vdash A0q_111 \vdash \)
Example 1: TM for $\{0^n1^n \mid n > 0\}$

- Accepts input 0011: $q_00011 \vdash Aq_1011 \vdash A0q_111 \vdash Aq_20B1 \vdash$
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Example 1: TM for $\{0^n1^n \mid n > 0\}$

Accepts input 0011: $q_0$0011 $\vdash$ A$q_1$011 $\vdash$ A0$q_1$11 $\vdash$ A$q_2$0$B_1$ $\vdash$ q$_2$A0$B_1$ $\vdash$ A$q_0$0$B_1$ $\vdash$ AA$q_1$B1 $\vdash$ AAB$q_1$1 $\vdash$ AA$q_2$BB $\vdash$ A$q_2$ABB $\vdash$
Example 1: TM for \( \{0^n1^n \mid n > 0\} \)

- Accepts input 0011: \( q_0 \)0011 \( \vdash \) \( Aq_1011 \) \( \vdash \) \( A0q_111 \) \( \vdash \) \( Aq_20B1 \) \( \vdash \) \( q_2A0B1 \) \( \vdash \) \( Aq_00B1 \) \( \vdash \) \( AAq_1B1 \) \( \vdash \) \( AABq_11 \) \( \vdash \) \( AAq_2BB \) \( \vdash \) \( Aq_2ABB \) \( \vdash \) \( AAq_0BB \) \( \vdash \)
Example 1: TM for \( \{0^n1^n \mid n > 0\} \)

- **Accepts input 0011:**
  
  \[
  q_0 \xrightarrow{0} A, R \quad \xrightarrow{1} B, L \quad \xrightarrow{\sqcap} \sqcap, R \\
  B \xrightarrow{0} 0, R \quad \xrightarrow{B} B, R \quad 0 \xrightarrow{0, L} \\
  B \xrightarrow{B} B, R
  \]

  \[
  q_0 \xrightarrow{0011} Aq_1011 \xrightarrow{1} Aq_111 \xrightarrow{B} q_20B1 \xrightarrow{1} q_2A0B1 \xrightarrow{0} q_2B0B1 \xrightarrow{1} AAq_1B1 \xrightarrow{1} AABq_11 \xrightarrow{B} AAq_2BB \xrightarrow{1} Aq_2ABB \xrightarrow{B} AAq_0BB \xrightarrow{1} AABq_3B
  \]
Example 1: TM for \( \{0^n1^n \mid n > 0\} \)

![Diagram of a Turing machine](image)

Accepts input 0011: \( q_00011 \vdash Aq_1011 \vdash A0q_111 \vdash Aq_20B1 \vdash q_2A0B1 \vdash Aq_00B1 \vdash AAq_1B1 \vdash AABq_11 \vdash AAq_2BB \vdash Aq_2ABB \vdash AAq_0BB \vdash AABq_3B \vdash AABBq_3\triangleleft \vdash \)
Example 1: TM for $\{0^n1^n \mid n > 0\}$

Accepts input 0011: $q_00011 \vdash Aq_1011 \vdash A0q_111 \vdash Aq_20B1 \vdash q_2A0B1 \vdash Aq_00B1 \vdash AAq_1B1 \vdash AABq_11 \vdash AAq_2BB \vdash Aq_2ABB \vdash AAq_0BB \vdash AABq_3B \vdash AABBBq_3 \sqcup \vdash AABBB \sqcup q_{\text{acc}} \sqcup$
Example 1: TM for $\{0^n1^n \mid n > 0\}$

- **Accepts input 0011:** $q_00011 \vdash Aq_1011 \vdash A0q_111 \vdash Aq_20B1 \vdash q_2A0B1 \vdash Aq_00B1 \vdash AAq_1B1 \vdash AABq_11 \vdash AAq_2BB \vdash Aq_2ABB \vdash AAq_0BB \vdash AABq_3B \vdash AABBq_3 \sqcup \vdash AABB\sqcup q_{acc}\sqcup$

- **Rejects input 00:** $q_000 \vdash Aq_10 \vdash A0q_1 \sqcup \vdash$
Example 1: TM for \( \{0^n1^n \mid n > 0\} \)

- **Accepts input 0011:** 
  - \( q_0 \) 0011 ⊢ \( Aq_1 \) 011 ⊢ \( Aq_2 \) B1 ⊢ \( q_2 \) A0B1 ⊢ \( Aq_0 \) 0B1 ⊢ \( AAq_1 \) B1 ⊢ \( AABq_1 \) 1 ⊢ \( AAq_2 \) BB ⊢ \( Aq_2 \) ABB ⊢ \( AAq_0 \) BB ⊢ \( AABq_3 \) B ⊢ \( AABBq_3 \) ⊢ \( AABB \) \( \sqcup \) ⊢ \( q_{\text{acc}} \) \( \sqcup \)

- **Rejects input 00:** 
  - \( q_0 \) 00 ⊢ \( Aq_1 \) 0 ⊢ \( A0q_1 \) ⊢ \( A0 \) \( \sqcup \) ⊢ \( q_{\text{rej}} \) \( \sqcup \)
Example: \( \{0^n1^n \mid n > 0\} \)

Formal Definition

The machine is \( M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}}) \) where

\[
\begin{align*}
\delta(q_0, 0) &= (q_1, A, R) \\
\delta(q_0, B) &= (q_3, B, R) \\
\delta(q_1, 0) &= (q_1, 0, R) \\
\delta(q_1, B) &= (q_1, B, R) \\
\delta(q_1, 1) &= (q_2, B, L) \\
\delta(q_2, B) &= (q_2, B, L) \\
\delta(q_2, 0) &= (q_2, 0, L) \\
\delta(q_2, A) &= (q_0, A, R) \\
\delta(q_3, B) &= (q_3, B, R) \\
\delta(q_3, \sqcup) &= (q_{\text{acc}}, \sqcup, R) \\
\text{in all other cases, } \delta(q, X) &= (q_{\text{rej}}, \sqcup, R).
\end{align*}
\]
Example: \( \{0^n1^n \mid n > 0\} \)

Formal Definition

The machine is \( M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}}) \) where

- \( Q = \{q_0, q_1, q_2, q_3, q_{\text{acc}}, q_{\text{rej}}\} \)
Example: $\{0^n1^n \mid n > 0\}$

Formal Definition

The machine is $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$ where

- $Q = \{q_0, q_1, q_2, q_3, q_{\text{acc}}, q_{\text{rej}}\}$
- $\Sigma = \{0, 1\}$, and
Example: \( \{0^n1^n \mid n > 0\} \)

Formal Definition

The machine is \( M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}}) \) where

- \( Q = \{q_0, q_1, q_2, q_3, q_{\text{acc}}, q_{\text{rej}}\} \)
- \( \Sigma = \{0, 1\} \), and \( \Gamma = \{0, 1, A, B, \square\} \)
Example: \( \{0^n1^n \mid n > 0 \} \)

Formal Definition

The machine is \( M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}}) \) where

- \( Q = \{q_0, q_1, q_2, q_3, q_{\text{acc}}, q_{\text{rej}}\} \)
- \( \Sigma = \{0, 1\}, \) and \( \Gamma = \{0, 1, A, B, \sqcup\} \)
- \( \delta \) is given as follows

\[
\begin{align*}
\delta(q_0, 0) &= (q_1, A, R) & \delta(q_0, B) &= (q_3, B, R) \\
\delta(q_1, 0) &= (q_1, 0, R) & \delta(q_1, B) &= (q_1, B, R) \\
\delta(q_1, 1) &= (q_2, B, L) & \delta(q_2, B) &= (q_2, B, L) \\
\delta(q_2, 0) &= (q_2, 0, L) & \delta(q_2, A) &= (q_0, A, R) \\
\delta(q_3, B) &= (q_3, B, R) & \delta(q_3, \sqcup) &= (q_{\text{acc}}, \sqcup, R)
\end{align*}
\]

In all other cases, \( \delta(q, X) = (q_{\text{rej}}, \sqcup, R) \).
Example: \( \{0^n1^n \mid n > 0 \} \)

Formal Definition

The machine is \( M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}}) \) where

- \( Q = \{ q_0, q_1, q_2, q_3, q_{\text{acc}}, q_{\text{rej}} \} \)
- \( \Sigma = \{0, 1\} \), and \( \Gamma = \{0, 1, A, B, \square\} \)
- \( \delta \) is given as follows

\[
\begin{align*}
\delta(q_0, 0) &= (q_1, A, R) & \delta(q_0, B) &= (q_3, B, R) \\
\delta(q_1, 0) &= (q_1, 0, R) & \delta(q_1, B) &= (q_1, B, R) \\
\delta(q_1, 1) &= (q_2, B, L) & \delta(q_2, B) &= (q_2, B, L) \\
\delta(q_2, 0) &= (q_2, 0, L) & \delta(q_2, A) &= (q_0, A, R) \\
\delta(q_3, B) &= (q_3, B, R) & \delta(q_3, \square) &= (q_{\text{acc}}, \square, R) \\
\end{align*}
\]

In all other cases, \( \delta(q, X) = (q_{\text{rej}}, \square, R) \). So for example, \( \delta(q_0, 1) = (q_{\text{rej}}, \square, R) \).
Example 2: TM for \( \{0^n1^n2^n \mid n > 0\} \)

Design a TM to accept the language \( L = \{0^n1^n2^n \mid n > 0\} \)
Example 2: TM for \( \{0^n1^n2^n \mid n > 0\} \)

Design a TM to accept the language \( L = \{0^n1^n2^n \mid n > 0\} \)

High level description

On input string \( w \)

while there are unmarked 0s, do
  Mark the left most 0
  Scan right to reach the leftmost unmarked 1;
  if there is no such 1 then crash
  Mark the leftmost 1
  Scan right to reach the leftmost unmarked 2;
  if there is no such 2 then crash
  Mark the leftmost 2
done

Check to see that there are no unmarked 1s or 2s;
  if there are then then crash
accept
Example 2: TM for \( \{0^n1^n2^n \mid n > 0\} \)
Example 2: TM for \( \{0^n1^n2^n \mid n > 0\} \)

\[
\begin{align*}
q_0 \xrightarrow{0} & A, R \\
q_0 \xrightarrow{B} & B, R \\
q_4 \xrightarrow{=} & =, R \\
q_{\text{acc}} \xrightarrow{0} & 0, L \\
q_3 \xrightarrow{C} & C, R
\end{align*}
\]

\[
\begin{align*}
q_0 \xrightarrow{1} & B, R \\
q_2 \xrightarrow{1} & C, R \\
q_1 \xrightarrow{B} & B, R \\
q_2 \xrightarrow{C} & C, R \\
q_1 \xrightarrow{\text{⊔}} & \text{⊔}, R \\
q_4 \xrightarrow{\text{⊔}} & \text{⊔}, R \\
q_3 \xrightarrow{\text{⊔}} & \text{⊔}, R
\end{align*}
\]

\[
\begin{align*}
q_0 \xrightarrow{2} & C, L \\
q_3 \xrightarrow{2} & C, L \\
q_1 \xrightarrow{A} & A, R \\
q_0 \xrightarrow{A} & A, R \\
q_2 \xrightarrow{B} & B, R \\
q_2 \xrightarrow{C} & C, R \\
q_1 \xrightarrow{\text{⊔}} & \text{⊔}, R \\
q_4 \xrightarrow{\text{⊔}} & \text{⊔}, R
\end{align*}
\]

\[
e.g.:\ q_0001122\vdash^* A0Bq_31C2
\]
Example 2: TM for $\{0^n1^n2^n \mid n > 0\}$

\begin{align*}
&\text{q}_0 & \rightarrow & \begin{array}{c}
0 \rightarrow A, R \\
B \rightarrow B, R \\
\square \rightarrow \square, R \\
B \rightarrow B, R \\
C \rightarrow C, R
\end{array}
\\&\text{q}_1 & \rightarrow & \begin{array}{c}
1 \rightarrow B, R \\
0 \rightarrow 0, R \\
B \rightarrow B, R
\end{array}
\\&\text{q}_2 & \rightarrow & \begin{array}{c}
2 \rightarrow C, L \\
1 \rightarrow 1, R \\
C \rightarrow C, R
\end{array}
\\&\text{q}_3 & \rightarrow & \begin{array}{c}
C \rightarrow C, L \\
1 \rightarrow 1, L \\
B \rightarrow B, L \\
0 \rightarrow 0, L
\end{array}
\\&\text{q}_4 & \rightarrow & \begin{array}{c}
\square \rightarrow \square, R \\
B \rightarrow B, R \\
C \rightarrow C, R
\end{array}
\\&\text{q}_{\text{acc}} & \rightarrow & A \rightarrow A, R
\end{align*}

e.g.: $q_0001122 \vdash^* A0Bq_31C2 \vdash^* q_3A0B1C2$
Example 2: TM for \( \{0^n1^n2^n | n > 0\} \)

\[
\begin{align*}
q_0 \rightarrow & A, R \\
& 0 \rightarrow A, R \\
q_1 \rightarrow & B, R \\
& 1 \rightarrow B, R \\
q_2 \rightarrow & C, L \\
& 2 \rightarrow C, L \\
q_3 \rightarrow & \text{acc} \\
& C \rightarrow C, L \\
& 1 \rightarrow 1, L \\
& B \rightarrow B, L \\
q_4 \rightarrow & \text{acc} \\
& B \rightarrow B, R \\
& C \rightarrow C, R \\
\end{align*}
\]

\[
\text{e.g.: } q_0001122 \vdash *A0Bq_31C2 \vdash *q_3A0B1C2 \vdash Aq_00B1C2
\]
Example 2: TM for $\{0^n1^n2^n \mid n > 0\}$

e.g.: $q_0001122 \vdash * A0Bq_31C2 \vdash * q_3A0B1C2 \vdash Aq_00B1C2 \vdash * AAq_0BBCC$
Example 2: TM for \( \{0^n1^n2^n \mid n > 0\} \)

e.g.: \( q_0001122 \vdash *A0Bq_31C2 \vdash *q_3A0B1C2 \vdash Aq_00B1C2 \vdash *AAq_0BBCC \vdash *AABBCcq_4 \)
Example 2: TM for $\{0^n1^n2^n \mid n > 0\}$

e.g.: $q_0001122 \vdash * A0Bq_31C2 \vdash * q_3A0B1C2 \vdash Aq_00B1C2$
$\vdash * AAq_0BBCC \vdash * AABBCCq_4 \sqsubseteq \vdash AABBCC \sqsubseteq q_{\text{acc}} \sqsubseteq$
Recognizing/Deciding a Language

▶ Only halting configurations are those with state $q_{\text{acc}}$ or $q_{\text{rej}}$.

▶ A Turing machine may keep running forever on some input.

▶ Then the machine does not accept that input.

▶ So two ways to not accept: reject or never halt (loop).

Three possible outcomes: accept, reject, or loop.
Recognizing/Deciding a Language

- Only halting configurations are those with state $q_{\text{acc}}$ or $q_{\text{rej}}$
Recognizing/Deciding a Language

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Recognizing/Deciding a Language

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Recognizing/Deciding a Language

- Only halting configurations are those with state $q_{\text{acc}}$ or $q_{\text{rej}}$
- A Turing machine may keep running forever on some input
- Then the machine does not accept that input
- So two ways to not accept: reject or never halt (loop)
Recognizing/Deciding a Language

- Only halting configurations are those with state $q_{\text{acc}}$ or $q_{\text{rej}}$
- A Turing machine may keep running forever on some input
- Then the machine does not accept that input
- So two ways to not accept: reject or never halt (loop)
- Three possible outcomes: accept, reject, or loop
Recognizing a Language

Definition
A language $L$ is said to be Turing-recognizable if there is some Turing machine $M$ that recognizes it.
That is, for every $w \in L$, $M$ must accept $w$. Also for every $w \in L$, $M$ must not accept $w$ (either reject it or does not halt).
Deciding a Language

Definition
A Turing machine $M$ is said to decide a language $L$ if $L = L(M)$ and $M$ halts on every input.

Definition
A language $L$ is said to be Turing-decidable (or simply decidable) if there is some Turing machine $M$ that decides it.
Deciding a Language

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Deciding a language is more than recognizing it!
Deciding a Language

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A Turing machine $M$ is said to decide a language $L$ if $L = L(M)$ and $M$ halts on every input.

Definition
A language $L$ is said to be Turing-decidable (or simply decidable) if there is some Turing machine $M$ that decides it.

Deciding a language is more than recognizing it! There are languages which are recognizable, but not decidable.