# CSE 135: Introduction to Theory of Computation
# Context-Free Languages and Ambiguity

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Conventions.
$V$: uppercase; $\Sigma$: lowercase, numbers, special symbols; $S$: Var on the LHS of the topmost rule.

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Or more briefly, $R = \{S \to \epsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1\}$

# Example: Palindromes

Can you tell what are variables, terminals, and the start symbol?

**Example**

$R = \{S \to \epsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1\}$

# Language of a CFG

## Derivations

Expand the start symbol using one of its rules. Further expand the resulting string by expanding one of the variables in the string, by the RHS of one of its rules. Repeat until you get a string of terminals.

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$$S \Rightarrow 0S0 \Rightarrow 00S00 \Rightarrow 001S100 \Rightarrow 0010100$$

# Formal Definition

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Let $G = (V, \Sigma, R, S)$ be a CFG. We say $\alpha A \beta \Rightarrow_G \alpha\gamma\beta$, where $\alpha, \beta, \gamma \in (V \cup \Sigma)^*$ and $A \in V$ if $A \rightarrow \gamma$ is a rule of $G$.

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We say $\alpha \overset{*}{\Rightarrow}_G \beta$ if either $\alpha = \beta$ or there are $\alpha_0, \alpha_1, \ldots \alpha_n$ such that

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## Notation

When $G$ is clear from the context, we will write $\Rightarrow$ and $\overset{*}{\Rightarrow}$ instead of $\Rightarrow_G$ and $\overset{*}{\Rightarrow}_G$.

# Formal Definition

**Example**

For the given CFG $R = \{S \to aSb \mid SS \mid \epsilon\}$, show a derivation of strings $abab, aababb$.

# Design CFGs

### Example

Give a grammar for the language $\{0^n1^n \mid n \geq 0\} \cup \{1^n0^n \mid n \geq 0\}$

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$$\begin{array}{rcl} S & \to & S_1 \mid S_2 \\ S_1 & \to & 0S_11 \mid \epsilon \\ S_2 & \to & 1S_20 \mid \epsilon \end{array}$$

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$$S \rightarrow S00 \mid S01 \mid S10 \mid S11 \mid \epsilon$$

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\begin{aligned}
S &\rightarrow A111 \\
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# Context-Free Language

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The language of CFG $G = (V, \Sigma, R, S)$, denoted $L(G)$ is the collection of strings over the terminals derivable from $S$ using the rules in $R$. In other words,

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## Definition

A language $L$ is said to be context-free if there is a CFG $G$ such that $L = L(G)$.

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Recall, $L_{\mathrm{pal}} = \{w \in \{0,1\}^* \mid w = w^R\}$ is the language of palindromes.

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## Proposition

$L(G_{\rm pal}) = L_{\rm pal}$

# Proving Correctness of CFG

## $L_{\mathrm{pal}} \subseteq L(G_{\mathrm{pal}})$

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### Proof (contd).

Let $w \in L(G)$, i.e., $S \overset{*}{\Rightarrow} w$. We will show $w \in L_{\mathrm{pal}}$

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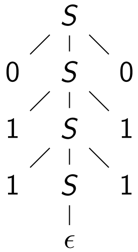
# Parse Trees

For CFG $G = (V, \Sigma, R, S)$, a parse tree (or derivation tree) of $G$ is a tree satisfying the following conditions:




Example Parse Tree with yield 011110

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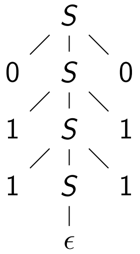


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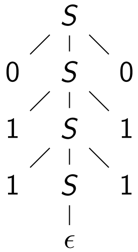


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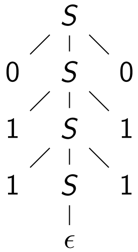
- If an interior node labeled by $A$ with children labeled by $X_1, X_2, \ldots X_k$ (from the left), then $A \to X_1 X_2 \cdots X_k$ must be a rule.

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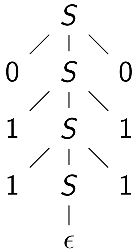


Example Parse Tree with yield 011110

Yield of a parse tree is the concatenation of leaf labels (left–right)

# Parse Trees and Derivations

### Proposition

*Let $G = (V, \Sigma, R, S)$ be a CFG. For any $A \in V$ and $\alpha \in (V \cup \Sigma)^*$, $A \stackrel{*}{\Rightarrow} \alpha$ iff there is a parse tree with root labeled $A$ and whose yield is $\alpha$.*

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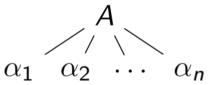
## Proof.

($\Rightarrow$): Proof by induction on the number of steps in the derivation.

- Base Case: If $A \Rightarrow \alpha$ then $A \to \alpha$ is a rule in $G$. There is a tree of height 1, with root $A$ and leaves the symbols in $\alpha$.




Parse Tree for Base Case

# Parse Trees for Derivations

Proof (contd).

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- Induction Step: Let $A \overset{*}{\Rightarrow} \alpha$ in $k+1$ steps.

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- Then $A \stackrel{*}{\Rightarrow} \alpha_1 X \alpha_2 \Rightarrow \alpha_1 \gamma \alpha_2 = \alpha$, where $X \rightarrow X_1 \cdots X_n = \gamma$ is a rule

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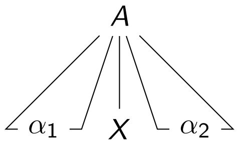
- By ind. hyp., there is a tree with root $A$ and yield $\alpha_1 X \alpha_2$.



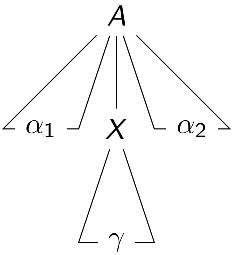

Parse Tree for Induction Step

# Parse Trees for Derivations

## Proof (contd).

(⇒): Proof by induction on the number of steps in the derivation.

- Induction Step: Let $A \overset{*}{\Rightarrow} \alpha$ in $k+1$ steps.
- Then $A \overset{*}{\Rightarrow} \alpha_1 X \alpha_2 \Rightarrow \alpha_1 \gamma \alpha_2 = \alpha$, where $X \rightarrow X_1 \cdots X_n = \gamma$ is a rule
- By ind. hyp., there is a tree with root $A$ and yield $\alpha_1 X \alpha_2$.
- Add leaves $X_1, \ldots X_n$ and make them children of $X$. New tree is a parse tree with desired yield. $\cdots\!\rightarrow$




Parse Tree for Induction Step

# Derivations for Parse Trees

Proof (contd).

($\Leftarrow$): Assume that there is a parse tree with root $A$ and yield $\alpha$. Need to show that $A \overset{*}{\Rightarrow} \alpha$.

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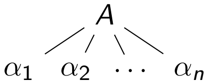
- Base Case: If tree has only one internal node, then it has the form as in picture




Parse Tree with one internal node

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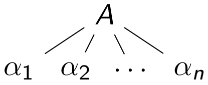
# Derivations for Parse Trees

## Proof (contd).

(⇐): Assume that there is a parse tree with root $A$ and yield $\alpha$. Need to show that $A \stackrel{*}{\Rightarrow} \alpha$. Proof by induction on the number of internal nodes in the tree.

- Base Case: If tree has only one internal node, then it has the form as in picture
- Then, $\alpha = X_1 \cdots X_n$ and $A \to \alpha$ is a rule. Thus, $A \stackrel{*}{\Rightarrow} \alpha$.

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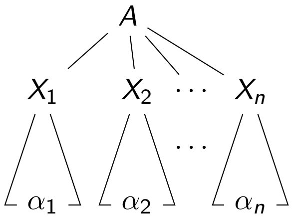
(⇐) Induction Step: Suppose $\alpha$ is the yield of a tree with $k+1$ interior nodes. Let $X_1, X_2, \ldots X_n$ be the children of the root ordered from the left. Not all $X_i$ are leaves, and $A \to X_1 X_2 \cdots X_n$ must be a rule.




Tree with $k+1$ internal nodes

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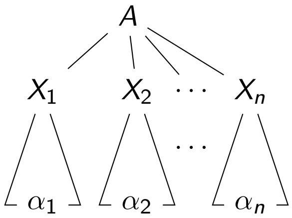
- Let $\alpha_i$ be the yield of the tree rooted at $X_i$; so $X_i$ is a leaf $\alpha_i = X_i$

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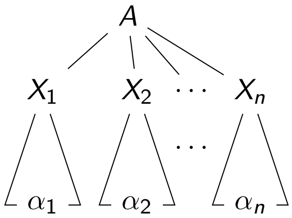
# Derivations for Parse Trees

## Proof (contd).

($\Leftarrow$) Induction Step: Suppose $\alpha$ is the yield of a tree with $k+1$ interior nodes. Let $X_1, X_2, \ldots X_n$ be the children of the root ordered from the left. Not all $X_i$ are leaves, and $A \to X_1 X_2 \cdots X_n$ must be a rule.

- Let $\alpha_i$ be the yield of the tree rooted at $X_i$; so $X_i$ is a leaf $\alpha_i = X_i$
- Now if $j < i$ then all the descendents of $X_j$ are to the left of the descendents of $X_i$. So $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$.




Tree with $k+1$ internal nodes

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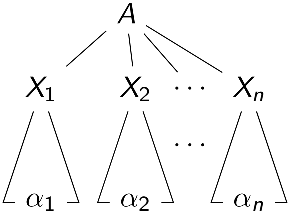

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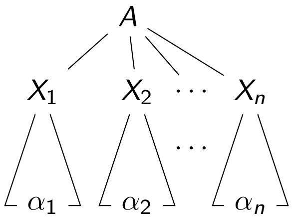
(⇐) Induction Step: Suppose $\alpha$ is the yield of a tree with $k + 1$ interior nodes.

- Each subtree rooted at $X_i$ has at most $k$ internal nodes. So if $X_i$ is a leaf $X_i \stackrel{*}{\Rightarrow} \alpha_i$ and if $X_i$ is not a leaf then $X_i \stackrel{*}{\Rightarrow} \alpha_i$ (ind. hyp.).



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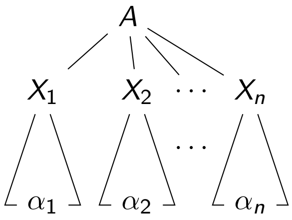
- Thus $A \Rightarrow X_1X_2\cdots X_n \stackrel{*}{\Rightarrow} \alpha_1X_2\cdots X_n \stackrel{*}{\Rightarrow} \alpha_1\alpha_2\cdots X_n \stackrel{*}{\Rightarrow} \alpha_1\cdots\alpha_n = \alpha$ $\square$



# Recap . . .

For a CFG $G$ with variable $A$ the following are equivalent

1. $A \stackrel{*}{\Rightarrow} w$
2. There is a parse tree with root $A$ and yield $w$

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## Context-free-ness

CFGs have the property that if $X \stackrel{*}{\Rightarrow} \gamma$ then $\alpha X \beta \stackrel{*}{\Rightarrow} \alpha \gamma \beta$

# Example: English Sentences

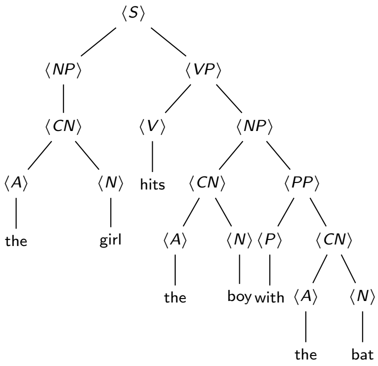
English sentences can be described as

$$
\begin{aligned}
\langle S \rangle &\rightarrow \langle NP \rangle \langle VP \rangle \\
\langle NP \rangle &\rightarrow \langle CN \rangle \mid \langle CN \rangle \langle PP \rangle \\
\langle VP \rangle &\rightarrow \langle CV \rangle \mid \langle CV \rangle \langle PP \rangle \\
\langle PP \rangle &\rightarrow \langle P \rangle \langle CN \rangle \\
\langle CN \rangle &\rightarrow \langle A \rangle \langle N \rangle \\
\langle CV \rangle &\rightarrow \langle V \rangle \mid \langle V \rangle \langle NP \rangle \\
\langle A \rangle &\rightarrow \text{a} \mid \text{the} \\
\langle N \rangle &\rightarrow \text{boy} \mid \text{girl} \mid \text{bat} \\
\langle V \rangle &\rightarrow \text{hits} \mid \text{likes} \mid \text{sees} \\
\langle P \rangle &\rightarrow \text{with}
\end{aligned}
$$

# Multiple Parse Trees

## Example 1

The sentence “the girl hits the boy with the bat” has the following parse tree



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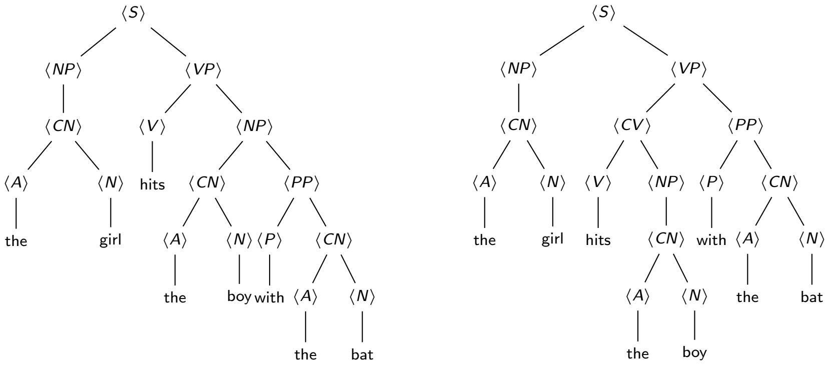

# Example: Arithmetic Expressions

Consider the language of all arithmetic expressions ($E$) built out of integers ($N$) and identifiers ($I$), using only $+$ and $*$

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$G_{\mathrm{exp}} = (\{E, I, N\}, \{a, b, 0, 1, (, ), +, *, -\}, R, E)$ where $R$ is

$$
\begin{aligned}
&E \rightarrow I \mid N \mid -N \mid E+E \mid E*E \mid (E) \\
&I \rightarrow a \mid b \mid Ia \mid Ib \\
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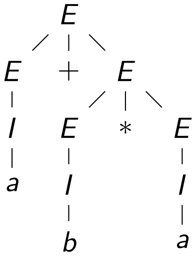
The parse tree for expression $a + b * a$ in the grammar $G_{\text{exp}}$ is



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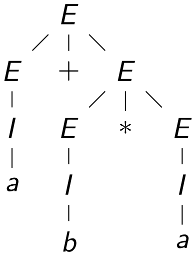


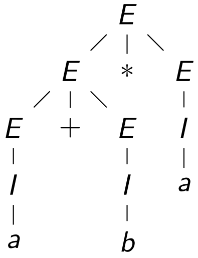

# Ambiguity

## Definition

A grammar $G = (V, \Sigma, R, S)$ is said to be ambiguous if there is $w \in \Sigma^*$ for which there are two different parse trees.

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### Warning!

Existence of two derivations for a string does not mean the grammar is ambiguous!

# Removing Ambiguity

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- Using the semantics to change the rules. For example, if we knew who had the bat (the girl or the boy) from the context, we would know which is the right interpretation.
- Adding precedence to operators. For example, $*$ binds more tightly than $+$, or "else" binds with the innermost "if".

# An Example

Recall, $G_{\mathrm{exp}}$ has the following rules

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&E \rightarrow I \mid N \mid -N \mid E + E \mid E * E \mid (E)\\
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New CFG $G'_{\mathrm{exp}}$ has the rules

$$I \to a \mid b \mid Ia \mid Ib$$
$$N \to 0 \mid 1 \mid N0 \mid N1$$
$$F \to I \mid N \mid -N \mid (E)$$
$$T \to F \mid T*F$$
$$E \to T \mid E+T$$

# Ambiguity: Computational Problems

### Removing Ambiguity

Problem: Given CFG $G$, find CFG $G'$ such that $L(G) = L(G')$ and $G'$ is unambiguous.

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Problem: Given CFG $G$, determine if $G$ is ambiguous.

The problem is undecidable.

Problem: Is it the case that for every CFG $G$, there is a grammar $G'$ such that $L(G) = L(G')$ and $G'$ is unambiguous, *even if* $G'$ *cannot be constructed algorithmically*?

# Inherently Ambiguous Languages

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No! There are context-free languages $L$ such that every grammar for $L$ is ambiguous.

## Definition

A context-free language $L$ is said to be inherently ambiguous if every grammar $G$ for $L$ is ambiguous.

# Inherently Ambiguous Languages

## An Example

Consider

$$L = \{a^i b^j c^k \mid i = j \text{ or } j = k\}$$

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